

Symanzik effective actions in the large N limit

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ABSTRACT: Symanzik effective actions, conjectured to describe lattice artifacts, are determined for a class of lattice regularizations of the non-linear $O(N)$ sigma model in two dimensions in the leading order of the $1/N$ -expansion. The class of actions considered includes also ones which do not have the usual classical limit and are not (so far) treatable in the framework of ordinary perturbation theory. The effective actions obtained are shown to reproduce previously computed lattice artifacts of the step scaling functions defined in finite volume, giving further confidence in Symanzik's theory of lattice artifacts.

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1 Introduction

Determinations of physical quantities in lattice QCD require extrapolation of the data to the continuum limit. For this purpose an ansatz must be made on the nature of the lattice artifacts. Usually these are assumed to be predominantly power-like $\sim a^p$ in the lattice spacing a , the power p depending on the particular lattice action used for the simulation.

The latter ansatz is based on a *conjecture* of Symanzik [1] which says that the lattice artifacts of correlation functions of the basic fields are described by a local effective continuum Lagrangian involving higher dimensional operators which may have only the symmetries of the underlying lattice theory, and coefficients depending on a ¹.

¹The description of lattice artifacts of correlation functions involving composite operators involve in addition further effective operators.

The study of lattice artifacts is not only of theoretical interest: it is closely related to Symanzik’s improvement program [2–5] where the leading cutoff effects are cancelled by adding (already on the lattice) $\mathcal{O}(a^2)$ local counterterms with suitably fine-tuned coefficients. Using such improved actions the lattice artifacts are reduced considerably and since the continuum limit is reached faster, allows for cheaper MC simulations.

This structural conjecture is based mainly on perturbative computations, both in the coupling and in the $1/N$ expansion, in various models. An all order proof for the existence of Symanzik effective actions for ϕ_4^4 theory and QED_4 has been given by Keller [6], albeit this proof is only in the framework of a continuum regularization. However, these theories, at least when regularized on the lattice, are probably trivial in the continuum limit. Renormalized couplings tend to zero as $g \sim c/\ln(a\mu)$ and hence the continuum limit (a free theory) is actually reached only logarithmically! Treating a renormalized coupling effectively as a constant for a range of cutoffs one has for small g a perturbative Lagrangian description of the low energy physics, and in this case the Symanzik effective Lagrangian describes the leading a^2 cutoff corrections to this, in particular effects of rotation symmetry breaking.

So far an all order perturbative proof of the conjecture has not been given for asymptotically free theories; there seems to be no technical obstacle but it is probably a tedious task. For QCD there is support for the validity of the conjecture coming only from low order perturbative computations, and from the fact that the numerical data can be convincingly described by fits of the expected form.

Asymptotically free theories in 2 dimensions such as the $O(N)$ sigma model, offer a laboratory to study cutoff effects in more detail, since simulations to large correlation lengths can be made, and one can also perform studies in the $1/N$ -expansion. We shall restrict attention to this theory in this paper and the structure of the Symanzik effective action is discussed in the next subsection. For a class of lattice actions permitting a standard perturbative formulation it was shown [7] that the generic leading cutoff behavior is $\propto a^2(\ln ma)^{N/(N-2)}$ (see subsection 1.2). The additional $\ln^3 a$ factor for $N = 3$ becomes visible in precision measurements, and can even mimic $\mathcal{O}(a)$ effects in some intermediate range of correlation lengths.

In a recent paper [8] a class of models was identified with very small lattice artifacts. Some of these models are particularly fascinating since they seem to belong to the same universality class as the standard-type actions described above but don’t have the usual classical limit and so far a perturbative treatment for them is unknown (see subsection 1.2). However, they can be treated in the $1/N$ approximation and there the step scaling functions (of the finite volume mass gap) were found to be of the form $a^2 \ln^s a$ with $s = 2$ in leading order $1/N$ rather than $\lim_{N \rightarrow \infty} N/(N-2) = 1$. Note that in the framework of Symanzik’s conjecture, the difference between the cutoff behavior for different lattice actions can only arise from the couplings in the effective theory.

In this paper we determine the Symanzik effective action for the classes of lattice actions mentioned above in the leading order of the $1/N$ expansion by matching lattice correlation functions to the effective theory. We show how the constructed effective theory reproduces the artifacts of the step scaling functions defined in finite volume. It is instructive to see

how the additional non-perturbative effects leading to the $\ln^2 a$ appear in this framework. The various steps of the computation are given in sections 2-4 and some technical details are relegated to appendices.

1.1 Symanzik's effective action for $2d$ $O(N)$ sigma models

As explained above, Symanzik's idea is to mimic the lattice artifacts (at least up to the leading $\mathcal{O}(a^2)$ order) by using an effective Lagrangian in the continuum theory:

$$-\mathcal{L}_{\text{eff}} = -\mathcal{L} + a^2 \sum_i c_i(g) U_i, \quad (1.1)$$

where \mathcal{L} is the original continuum Lagrangian density and in this sum all operators U_i with engineering dimension 4 and reflecting the lattice symmetries have to be taken into account. The c-number coefficients $c_i(g)$ depend on the lattice action. For perturbative type actions they can be calculated in perturbation theory. In this paper we will calculate them in the large N expansion.

For practical calculations a useful starting point is the master formula of Symanzik's effective theory, which can be obtained using the effective Lagrangian (1.1) and is written directly in terms of correlation functions on the lattice and the corresponding effective continuum model:

$$\mathcal{G}_{\text{latt}}^X(\lambda_0, a) = y^r(g) \mathcal{G}_{(R)}^X(g, a^{-1}) + a^2 y^r(g) \sum_i v_i(g) \mathcal{G}_{i(R)}^X(g, a^{-1}) + \mathcal{O}(a^4). \quad (1.2)$$

Here $\mathcal{G}_{\text{latt}}^X$ represents a general lattice correlation function (with r external legs) for any physical quantity X , in real space or momentum space, and $\mathcal{G}_{(R)}^X$ is the analogous renormalized quantity calculated in the continuum model. $\mathcal{G}_{\text{latt}}^X$ depends on the set λ_0 of bare lattice couplings and parameters, while the continuum correlators depend on the renormalized coupling g and the renormalization scale μ , which is taken here, for simplicity, as the inverse lattice spacing a^{-1} . $\mathcal{G}_{i(R)}^X$ is the corresponding renormalized continuum correlation function calculated with insertion of the integral of one of the local dimension 4 operators, U_i , appearing in Symanzik's effective continuum action. For a given lattice symmetry the set of operators is fixed and only the c-number coefficient functions $v_i(g)$, the renormalized coefficients corresponding to $c_i(g)$, depend on the lattice action. Finally, the wave function renormalization constant $y(g)$ and the relation $g = g(\lambda_0)$ between the lattice coupling parameters and the continuum coupling constant are again action dependent. The latter can be obtained by calculating some physical mass parameter \mathcal{M} on both sides:

$$\mathcal{M}(g, a^{-1}) = \mathcal{M}_{\text{latt}}(\lambda_0). \quad (1.3)$$

For generating correlation functions we will use the source dependent action

$$\mathcal{A} = \int d^D x \mathcal{L} = \int d^D x \left\{ \frac{1}{2g_0^2} \partial_\mu S \cdot \partial_\mu S - \frac{1}{g_0^2} I \cdot S \right\}. \quad (1.4)$$

Here S is the N -component sigma model field with normalization $S \cdot S = 1$ and instead of the bare coupling constant g_0^2 we will mainly use the 't Hooft coupling $f_0 = N g_0^2$ (and similarly

for the renormalized quantities: $f = Ng^2$). In the continuum we will use dimensional regularization in $D = 2 - \varepsilon$ dimensions.

The set of local operators appearing in Symanzik's effective action always includes the following Lorentz-scalar dimension 4 $O(N)$ invariant operators:

$$\begin{aligned}\mathcal{O}_1 &= \frac{1}{8}(\partial_\mu S \cdot \partial_\mu S)^2, \\ \mathcal{O}_2 &= \frac{1}{8}(\partial_\mu S \cdot \partial_\nu S)(\partial_\mu S \cdot \partial_\nu S), \\ \mathcal{O}_3 &= \frac{1}{2}(\Box S \cdot \Box S).\end{aligned}\tag{1.5}$$

These operators mix under renormalization with other Lorentz-scalar operators and in Symanzik's effective action we also have to include operators which are not fully Lorentz scalar but invariant under lattice symmetries only. The full list of operators in the case of lattices with cubic symmetry will be given in section 2.

A set of lattice regularized $O(N)$ sigma models with one tunable coupling $\beta = 1/\lambda_0^2$ is given by the following quadratic lattice actions:

$$\mathcal{A}_{\text{latt}} = \frac{\beta}{2} a^4 \sum_{x,y} S(x) \cdot S(y) K(x-y).\tag{1.6}$$

Here the only requirement is that the Fourier transform of the inverse “propagator” $K(x)$ defined by

$$K_p = a^2 \sum_x e^{-iapx} K(x)\tag{1.7}$$

behaves for $a \rightarrow 0$ as

$$K_p = p^2 + \mathcal{O}(a^2).\tag{1.8}$$

This ensures that the lattice model has the correct classical continuum limit. For the standard lattice action (ST) we have:

$$K_p = \hat{p}^2, \quad \hat{p}_\mu = \frac{2}{a} \sin \frac{ap_\mu}{2}.\tag{1.9}$$

An alternative way of writing the master equation is

$$\mathcal{G}_{\text{latt}}^X = \mathcal{G}^{X(0)}(\lambda_0, a) \left\{ 1 + a^2 \delta^X(\lambda_0, a) + \mathcal{O}(a^4) \right\},\tag{1.10}$$

where the scaling part $\mathcal{G}^{X(0)}$ is universal (up to wave function renormalization) and the correction factor is of the form

$$\delta^X(\lambda_0, a) = \sum_i v_i(g) \delta_i^X(g, a).\tag{1.11}$$

Using perturbation theory and renormalization group considerations, it was shown in [7] that close to the continuum limit this correction can be represented as

$$\delta^X(\lambda_0, a) = C_1 D_1^X(\Lambda) \left\{ \tilde{\beta}^{1+2\chi} + \mathcal{O}(\tilde{\beta}^{2\chi}) \right\} + \mathcal{O}(\tilde{\beta}^\chi),\tag{1.12}$$

where $\chi = 1/(N - 2)$ and close to the continuum

$$\tilde{\beta} = \frac{2\pi}{\lambda_0} \sim \ln(\Lambda a). \quad (1.13)$$

In the above formula C_1 is a non-universal (action dependent) constant which is calculable in perturbation theory. On the other hand, $D_1^X(\Lambda)$ is universal but non-perturbative and depends on the quantity X in question and on the non-perturbative physical scale Λ . $D_1^X(\Lambda)$ is related to the non-perturbative matrix element of the operator with largest anomalous dimension. For $N = 3$ (1.12) predicts a large logarithmic correction proportional to $a^2(\ln a)^3$, whereas for large N this prediction gives $\mathcal{O}(a^2 \ln a)$ corrections. This is consistent with the known [9, 10] $\mathcal{O}(a^2 \ln a)$ corrections appearing for the standard lattice action in the large N limit.

1.2 Lattice artifacts for non-perturbative actions

The universality class of lattice regularized $O(N)$ sigma models is actually much larger than the set of quadratic actions (1.6). In [8] non-perturbative actions were studied. For these types of actions the classical continuum limit cannot be expanded by usual weak coupling perturbation theory or there is no classical limit at all! Nevertheless, as was demonstrated in [8] the exact quantum continuum limit of these models coincides with the usual one. Moreover, some of these models show much smaller lattice artifacts than the conventional ones described by (1.6) type actions.

The physical quantity studied in [8] was the step scaling function $\sigma(2, u)$ which is defined [11] as follows. Consider the model confined in a periodic (one-dimensional) box of size L and denote by $M(L)$ the mass gap in the box.² Introducing the dimensionless LWW coupling [11]

$$u = L M(L), \quad (1.14)$$

the step scaling function

$$\sigma(2, u) = 2L M(2L) \quad (1.15)$$

describes how the LWW coupling changes under doubling the size of the box (a discrete renormalization group transformation). On the lattice we have

$$u = L M(L, a), \quad \Sigma(2, u, a/L) = 2L M(2L, a). \quad (1.16)$$

We will also use the notations

$$u' = \Sigma(2, u, a/L), \quad u'_\infty = \sigma(2, u) = \Sigma(2, u, 0). \quad (1.17)$$

For large N the step scaling function is known exactly [12]. It is given by the solution of the implicit equation

$$f_0(u) = f_0(u'_\infty) + \frac{\ln 2}{2\pi}. \quad (1.18)$$

The function $f_0(u)$ will be given explicitly by (2.74).

² M or $M(L)$ always refers here the finite-volume mass gap, while we denote the infinite-volume mass gap by m .

Let us introduce the variable³

$$z = \frac{1}{2\pi} \ln \left(\frac{\sqrt{32}}{ma} \right). \quad (1.19)$$

($\xi = 1/(ma)$ is the correlation length in an infinite volume measured in lattice units.)

The $\mathcal{O}(a^2)$ lattice artifacts for the step scaling function can be written as

$$u' = u'_\infty + \frac{a^2}{L^2} \nu(u, z) + \mathcal{O}(a^4), \quad (1.20)$$

where the dependence of the correction coefficient $\nu(u, z)$ on a/L must be weak (logarithmic).

It was found in [8] that in terms of this variable the coefficient function $\nu(u, z)$ for the standard action is of the form⁴

$$\nu_{\text{ST}}(u, z) = t_0(u) + t_1(u) z. \quad (1.21)$$

This is consistent with the $\ln a$ behavior discussed above. For the non-perturbative constrained (con) and mixed (mix) lattice actions described in section 3 the corresponding coefficient functions are given by

$$\nu_{\text{con}}(u, z) = \bar{t}_0(u) + \bar{t}_1(u) z + \bar{t}_2(u) z^2, \quad (1.22)$$

and

$$\nu_{\text{mix}}(u, z) = T_0(u) + T_1(u) z + T_2(u) z^2, \quad (1.23)$$

respectively.

The $(\ln a)^2$ behavior found for the non-perturbative actions is in apparent contradiction with the result (1.12) found for quadratic (perturbative) actions. But one can notice (see section 3) the structure

$$\begin{aligned} t_1(u) &= -\frac{1}{8f'_0(u'_\infty)} \left(u^2 - \frac{1}{4} u'^2_\infty \right), \\ \bar{t}_2(u) &= -8t_1(u), \\ T_2(u) &\propto t_1(u). \end{aligned} \quad (1.24)$$

This structure hints at the fact that here $t_1(u)$, $\bar{t}_2(u)$ and $T_2(u)$ are all proportional to the matrix element of the same operator and only the behavior of the coefficient function (proportional to z for ST and to z^2 for con and mix) is different.

The purpose of this paper is to show by explicitly constructing the large N effective action and calculating the relevant matrix elements that the structure described above does indeed hold and more generally that Symanzik's effective action description is valid also beyond perturbation theory.

³In [8] it was defined as $z = f_0(u) + \ln(L/a)/(2\pi)$. Using eqs. (2.75) and (2.76) one can show that the two definitions coincide. Here we want to stress that z depends only on the lattice spacing and not on the physical volume.

⁴The functions $t_i(u)$ together with the similarly defined functions $\bar{t}_i(u)$ and $T_i(u)$ will be given in section 3.

1.3 Symanzik's strategy

To verify Symanzik’s effective action approach for large N $O(N)$ sigma models using the step scaling function we have to go through the following steps:

1. Calculate the coefficients in the master formula (1.2) by calculating the 2-point and 4-point correlation functions in infinite volume on both sides and comparing them. In the continuum we need the ordinary correlation functions and also correlation functions with dimension 4 operators appearing in the effective action inserted. On the lattice side we need the correlation functions expanded near the continuum limit up to $\mathcal{O}(a^2)$ precision.
2. Calculate the 2-point function (with and without operator insertion) in the continuum in finite volume. From this one can calculate the corrections to the mass gap and the step scaling function.
3. Compare the step scaling function directly calculated on the lattice with the ones obtained from Symanzik's effective action in step 2 using the coefficients calculated in step 1 and verify matching.

The dimensional regularization calculations necessary for steps 1 and 2 will be presented in section 2. The lattice calculations and the matching of the results with Symanzik’s effective theory will be presented in section 3 and section 4, respectively.

2 Symanzik theory for the large N sigma model

In this section we give the results for those continuum correlation functions of the $O(N)$ sigma model which are necessary to construct Symanzik's effective action in the large N limit. We will use dimensional regularization in $2 - \varepsilon$ dimensions.

We start by recalling the Feynman rules for the large N expansion for sigma models in Fourier space:

$S^a S^b$ (sigma field) propagator:

$$a \text{---} b \quad \frac{\delta^{ab}}{p^2 + m^2} \quad (2.1)$$

$\lambda\lambda$ (auxiliary field) propagator:

$$\text{wavy line} \quad \frac{-2}{NB(p)} \quad (2.2)$$

$S^a S^b \lambda$ vertex:

$$\delta^{ab} \quad (2.3)$$

Here the function $B(p)$ is given by the simple 1-loop integral

$$B(p) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)[m^2 + (k + p)^2]}. \quad (2.4)$$

Finally, correlation functions with r external sigma lines constructed by the above rules have to be multiplied by the overall factor

$$\left(\frac{f_0}{N}\right)^{r/2}, \quad (2.5)$$

and there is an extra 1/2 factor for every closed sigma loop.⁵ Only sigma loops with at least three $SS\lambda$ vertices need to be considered.

Next we give the complete list of local operators necessary to construct Symanzik's effective action for lattices with cubic symmetry. In addition to the Lorentz scalar operators \mathcal{O}_1 , \mathcal{O}_2 , \mathcal{O}_3 already defined in subsection 1.1, we need the following ones.

$$U_3 = -\frac{N}{4f_0} I \cdot \square S, \quad (2.6)$$

$$U_4 = \frac{N}{8f_0} (I \cdot S)^2 - \frac{I \cdot I}{8f_0}, \quad (2.7)$$

$$U_5 = \frac{N}{2f_0} I \cdot I, \quad (2.8)$$

$$U_6 = \frac{N}{f_0} \left\{ -\frac{6}{D+2} \mathcal{O}_3 + \sum_{\mu=1}^D S \cdot \partial_\mu^4 S \right\}, \quad (2.9)$$

$$U_7 = \frac{N}{f_0} \left\{ -\frac{8}{D+2} (\mathcal{O}_1 + 2\mathcal{O}_2) + \sum_{\mu=1}^D (\partial_\mu S \cdot \partial_\mu S)^2 \right\}, \quad (2.10)$$

$$U_8 = \frac{N}{4f_0} I \cdot S, \quad (2.11)$$

$$U_9 = \frac{N}{2f_0} (\partial_\mu S \cdot \partial_\mu S) - 2U_8. \quad (2.12)$$

We will use as our operator basis the operators U_3, \dots, U_9 just defined above and the combinations

$$U_1 = \frac{N}{f_0} \mathcal{O}_1 + \frac{1}{2} U_5 - U_3 - U_4, \quad U_2 = \frac{N}{f_0} (\mathcal{O}_1 - D\mathcal{O}_2). \quad (2.13)$$

U_3 , U_4 and U_5 are Lorentz-scalar but source-dependent operators, while U_6 and U_7 are components of traceless Lorentz-tensor operators invariant under the discrete (hyper-)cubic rotation symmetry group only. As explained in [7], it is necessary to include them in the set of operators in the effective action. Here the set is enlarged by the dimension 2 operators U_8 and U_9 . Since their matrix elements are proportional to the dynamically generated mass m^2 , they were not present in the perturbative treatment of [7], but need to be included here.

⁵Note that loops formed from the legs of our local operators do not count in this rule. All our diagrams shown in figures 1-10 are “tree” diagrams from this point of view and only the one corresponding to figure 11 is a one-loop diagram. The 1/2 rule applies for this last case only.

2-point functions

We will use the following notation for connected 2-point functions:

$$\langle S^a(x) S^a(y) \rangle_c = \mathcal{G}^{(2)}(x, y), \quad (2.14)$$

$$\langle K_i S^a(x) S^a(y) \rangle_c = \mathcal{G}_i^{(2)}(x, y). \quad (2.15)$$

Here K_i stands for the integrated (zero momentum) operator

$$K_i = \int d^D z U_i(z). \quad (2.16)$$

In Fourier space we define⁶

$$\int d^D x \int d^D y e^{ixq_1} e^{iyq_2} \mathcal{G}_i^{(2)}(x, y) = (2\pi)^D \delta(q_1 + q_2) \tilde{\mathcal{G}}_i^{(2)}(k), \quad (2.17)$$

where $q_1 = -q_2 = k$ and we will also use the “amputated” 2-point functions $G_i^{(2)}(k)$ defined by

$$\tilde{\mathcal{G}}_i^{(2)}(k) = \frac{G_i^{(2)}(k)}{(k^2 + m^2)^2}. \quad (2.18)$$

An important consequence of the normalization condition $S \cdot S = 1$ is that

$$\mathcal{G}^{(2)}(x, x) = 1 \quad (2.19)$$

and this can be rewritten in Fourier space as the gap equation to leading order in $1/N$

$$\frac{1}{f_0} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2}. \quad (2.20)$$

4-point functions

For connected 4-point functions we will use the analogous notations:

$$\langle S^a(x) S^a(y) S^b(w) S^b(z) \rangle_c = \mathcal{G}^{(4)}(x, y, w, z), \quad (2.21)$$

$$\langle K_i S^a(x) S^a(y) S^b(w) S^b(z) \rangle_c = \mathcal{G}_i^{(4)}(x, y, w, z), \quad (2.22)$$

$$\begin{aligned} \int d^D x \int d^D y \int d^D w \int d^D z e^{ixp_1} e^{iy p_2} e^{iw q_1} e^{iz q_2} \mathcal{G}_i^{(4)}(x, y, w, z) \\ = (2\pi)^D \delta(p_1 + p_2 + q_1 + q_2) \tilde{\mathcal{G}}_i^{(4)}(p_1, p_2, q_1, q_2), \end{aligned} \quad (2.23)$$

$$\tilde{\mathcal{G}}_i^{(4)}(p_1, p_2, q_1, q_2) = \frac{G_i^{(4)}(p_1, p_2, q_1, q_2)}{(p_1^2 + m^2)(p_2^2 + m^2)(q_1^2 + m^2)(q_2^2 + m^2)}. \quad (2.24)$$

⁶also similarly without the index i

Renormalization

We will have to perform the usual coupling constant and wave function renormalization, which in our problem has to be supplemented by the renormalization of local operators. We introduce the notation f for the renormalized 't Hooft coupling and $G_{i(R)}^{(r)}$, $G_{(R)}^{(r)}$ for renormalized (amputated) r -point functions with and without operator insertion. For the coupling renormalization we can use the gap equation (2.20) and we find

$$f_0 = \mu^\varepsilon Z(f, \varepsilon) f, \quad (2.25)$$

where

$$\frac{1}{Z} = 1 + \frac{f}{2\pi\varepsilon} + \mathcal{O}(1/N) \quad (2.26)$$

and

$$\frac{1}{f} = \frac{\gamma}{4\pi} + \frac{1}{2\pi} \ln \frac{\mu}{m}. \quad (2.27)$$

Here and below the constant γ is given by

$$\gamma = \ln 4\pi + \Gamma'(1). \quad (2.28)$$

The renormalized coupling depends on the renormalization scale μ , which we will eventually identify, for simplicity, with the inverse lattice spacing of the lattice model.

It turns out that the wave function renormalization constant Z is to leading order identical with the one appearing in (2.25) and (2.26) and all our operators U_i renormalize diagonally in the large N limit. If we define the operator renormalization constants Z_i by

$$G_{(R)}^{(r)} = Z^{-r/2} G^{(r)}, \quad G_{i(R)}^{(r)} = Z^{-r/2} Z_i G_i^{(r)} \quad (2.29)$$

we have

$$Z_1 = Z_9 = Z, \quad Z_2 = Z_4 = Z_7 = \frac{1}{Z}, \quad Z_3 = Z_5 = Z_6 = Z_8 = 1. \quad (2.30)$$

Renormalized 2-point functions

For the construction of Symanik's effective action we will need the dimensionally regularized, renormalized 2-point functions with and without insertion of the (integrated) local operators U_i . The details of the calculation will be presented in appendix A. Here we just

give a list of the results for the infinite volume renormalized 2-point functions.

$$G_{(R)}^{(2)} = f(k^2 + m^2), \quad (2.31)$$

$$G_{1(R)}^{(2)} = 2\pi m^4, \quad (2.32)$$

$$G_{2(R)}^{(2)} = 0, \quad (2.33)$$

$$G_{3(R)}^{(2)} = \frac{f}{2}k^2(k^2 + m^2), \quad (2.34)$$

$$G_{4(R)}^{(2)} = 0, \quad (2.35)$$

$$G_{5(R)}^{(2)} = f(k^2 + m^2)^2, \quad (2.36)$$

$$G_{6(R)}^{(2)} = 2f \left(k^4 - \frac{3}{4}(k^2)^2 \right), \quad (2.37)$$

$$G_{7(R)}^{(2)} = 0, \quad (2.38)$$

$$G_{8(R)}^{(2)} = \frac{f}{2}(k^2 + m^2), \quad (2.39)$$

$$G_{9(R)}^{(2)} = -4\pi m^2. \quad (2.40)$$

Here the notation

$$k^4 = \sum_{\mu=1,2} k_\mu^4 \quad (2.41)$$

is introduced for two-dimensional vectors.

Renormalized 4-point functions

The calculation of the renormalized 4-point functions will be presented in appendix B. Here we just list the results of the calculation. Below $p = p_1 + p_2 = -(q_1 + q_2)$.

$$G_{(R)}^{(4)} = \frac{-2f^2}{Nb(p)}, \quad (2.42)$$

$$G_{1(R)}^{(4)} = \frac{f}{Nb^2(p)} - \frac{m^2 f}{N} X, \quad (2.43)$$

$$G_{2(R)}^{(4)} = \frac{f^3}{N} \left\{ -Q^2 + Q \left(p_1 p_2 + q_1 q_2 - \frac{2(p_1 p)(p_2 p)}{p^2} - \frac{2(q_1 p)(q_2 p)}{p^2} \right) \right. \\ \left. + (p_1 p_2)(q_1 q_2) - (p_1 q_1)(p_2 q_2) - (p_1 q_2)(p_2 q_1) \right\}, \quad (2.44)$$

$$G_{3(R)}^{(4)} = -\frac{f^2}{2Nb(p)} (p_1^2 + p_2^2 + q_1^2 + q_2^2), \quad (2.45)$$

$$G_{4(R)}^{(4)} = \frac{f^3}{N} \left(\frac{p_1^2 + p_2^2}{2} + m^2 \right) \left(\frac{q_1^2 + q_2^2}{2} + m^2 \right), \quad (2.46)$$

$$G_{5(R)}^{(4)} = 0, \quad (2.47)$$

$$G_{6(R)}^{(4)} = -\frac{4f^2}{Nb(p)} \left\{ \frac{p_1^4}{p_1^2 + m^2} + \frac{p_2^4}{p_2^2 + m^2} + \frac{q_1^4}{q_1^2 + m^2} + \frac{q_2^4}{q_2^2 + m^2} \right\} \\ + \frac{3f^2}{Nb(p)} (p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{3m^2 f^2}{4\pi N} X + \frac{8f^2}{Nb^2(p)} Y, \quad (2.48)$$

$$G_{7(R)}^{(4)} = \frac{8f^3}{N} \left\{ Q^2 \left(-\frac{3}{4} + \frac{p^4}{(p^2)^2} \right) + Q \left(-\frac{p_1 p_2 + q_1 q_2}{4} - \frac{(p_1 p)(p_2 p) + (q_1 p)(q_2 p)}{2p^2} \right. \right. \\ \left. \left. + \sum_{\mu} \frac{p_{\mu}^2 (p_{1\mu} p_{2\mu} + q_{1\mu} q_{2\mu})}{p^2} \right) + \sum_{\mu} p_{1\mu} p_{2\mu} q_{1\mu} q_{2\mu} \right. \\ \left. - \frac{(p_1 p_2)(q_1 q_2) + (p_1 q_1)(p_2 q_2) + (p_1 q_2)(p_2 q_1)}{4} \right\}, \quad (2.49)$$

$$G_{8(R)}^{(4)} = -\frac{2f^2}{Nb(p)}, \quad (2.50)$$

$$G_{9(R)}^{(4)} = \frac{2f}{N} X. \quad (2.51)$$

Here we introduced the 2-dimensional limit of the (finite) loop integral (2.4):

$$b(p) = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(k^2 + m^2)[m^2 + (k + p)^2]}, \quad (2.52)$$

and used the notation

$$Q = m^2 - \frac{1}{4\pi b(p)}, \quad (2.53)$$

$$X = \frac{4\pi m^2}{b(p)} \left\{ \frac{1}{b(p)} \frac{\partial b(p)}{\partial m^2} + \frac{1}{p_1^2 + m^2} + \frac{1}{p_2^2 + m^2} + \frac{1}{q_1^2 + m^2} + \frac{1}{q_2^2 + m^2} \right\}, \quad (2.54)$$

$$\begin{aligned}
Y = & -3m^2b(p) + \frac{3p^2}{4} \left(m^2 + \frac{p^2}{4} \right) \frac{\partial b(p)}{\partial m^2} + \frac{3}{32\pi} \left(5 + \frac{p^2}{m^2} \right) \\
& + \frac{p^4}{(p^2)^2} \left\{ 2m^2b(p) - \left(\frac{m^4}{2} + m^2p^2 + \frac{(p^2)^2}{4} \right) \frac{\partial b(p)}{\partial m^2} - \frac{1}{8\pi} \left(5 + \frac{p^2}{m^2} \right) \right\}.
\end{aligned} \tag{2.55}$$

2-point function in strip geometry

After we determine the coefficients in Symanzik's effective action we can calculate the lattice correction to the mass gap and the step scaling function from the effective action if we know the 2-point functions (with and without operator insertions) in finite volume. In this subsection M is denoting the finite volume mass $M(L)$. (The mass parameter used in the infinite volume considerations is $m = M(\infty)$). In two dimensions space is finite (periodic with period L) in the 1 direction and infinite in the 2 (time) direction corresponding to a discrete spectrum of the momentum component k_1 and continuous spectrum for the momentum component k_2 . In D dimensions we will assume that the spectrum of k_1 is discrete and all other components k_2, \dots, k_D are continuous. We checked that the $D \rightarrow 2$ results remain the same if we assume that in D dimensions only the time component k_2 is continuous and all space components k_μ ($\mu = 1, 3, \dots, D$) are discrete. Finite volume quantities will be indicated by an overline. The details of the calculation can be found in appendix A. Here we only present the results.

$$\overline{G}_{(R)}^{(2)} = f(k^2 + M^2), \tag{2.56}$$

$$\overline{G}_{1(R)}^{(2)} = \frac{M^2}{2\overline{b}(0)}, \tag{2.57}$$

$$\overline{G}_{2(R)}^{(2)} = \frac{f^2 h}{2} \left(k_2^2 - k_1^2 + \frac{\ell}{\overline{b}(0)} \right), \tag{2.58}$$

$$\overline{G}_{3(R)}^{(2)} = \frac{f}{2} k^2 (k^2 + M^2), \tag{2.59}$$

$$\overline{G}_{4(R)}^{(2)} = 0, \tag{2.60}$$

$$\overline{G}_{5(R)}^{(2)} = f(k^2 + M^2)^2, \tag{2.61}$$

$$\overline{G}_{6(R)}^{(2)} = 2fk^4 + \frac{3f}{2} (M^4 - (k^2)^2) - \frac{2f\bar{\rho}}{\overline{b}(0)}, \tag{2.62}$$

$$\overline{G}_{2(R)}^{(7)} = f^2 h \left(k_1^2 - k_2^2 - \frac{\ell}{\overline{b}(0)} \right), \tag{2.63}$$

$$\overline{G}_{8(R)}^{(2)} = \frac{f}{2} (k^2 + M^2), \tag{2.64}$$

$$\overline{G}_{9(R)}^{(2)} = -\frac{1}{\overline{b}(0)}. \tag{2.65}$$

Here we introduced the finite volume integrals

$$h = \int_{(L)} \frac{d^2 q}{(2\pi)^2} \frac{q_1^2 - q_2^2}{q^2 + M^2} \quad \text{and} \quad \ell = \int_{(L)} \frac{d^2 q}{(2\pi)^2} \frac{q_1^2 - q_2^2}{(q^2 + M^2)^2}, \tag{2.66}$$

where the meaning of the integration symbol

$$\int_{(L)} \frac{d^2 q}{(2\pi)^2} \quad (2.67)$$

is an integration over the variable q_2 and sum over the quantized q_1 . The above two integrals are finite (after regularization). The parameter $\bar{\rho}$ is defined by a regularized (divergent) finite volume integral

$$\sum_{\mu=1}^D \int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + M^2)^2} + \frac{6M^2}{D+2} \frac{1}{f_0} = \bar{\rho} + \mathcal{O}(\varepsilon). \quad (2.68)$$

The finite volume gap equation and the step scaling function

The gap equation, which follows from the normalization $S \cdot S = 1$, is of the same form in finite volume as (2.20):

$$\frac{1}{f_0} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} = \int_{(L)} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + M^2}. \quad (2.69)$$

From this one can calculate [12] the (finite) relation between the infinite volume and finite volume mass parameters:

$$\ln \frac{m}{M} + F(u) = 0, \quad (2.70)$$

where $u = LM$ and

$$F(u) = \int_0^\infty \frac{dt}{t} e^{-tu^2} \left(\sum_{k=1}^\infty e^{-\frac{k^2}{4t}} \right). \quad (2.71)$$

An alternative representation of the function $F(u)$ is given by [9]

$$F(u) = \frac{\pi}{u} + \ln u - \gamma + G_0\left(\frac{u}{2\pi}\right), \quad (2.72)$$

where

$$G_0(w) = \sum_{k=1}^\infty \left\{ \frac{1}{\sqrt{k^2 + w^2}} - \frac{1}{k} \right\}. \quad (2.73)$$

We will often use the function⁷ $f_0(u)$ defined by

$$f_0(u) = \frac{1}{2\pi} \left\{ F(u) - \ln u + \ln \sqrt{32} \right\}, \quad (2.74)$$

in terms of which the step scaling transformation for a general scale s can be written as

$$f_0(u) = f_0(\sigma(s, u)) + \frac{\ln s}{2\pi}. \quad (2.75)$$

For further reference note that $f_0(u)$ has the asymptotic form [8]

$$f_0(u) = \frac{1}{2\pi} \left\{ -\ln u + \ln \sqrt{32} \right\} + \mathcal{O}(e^{-u}). \quad (2.76)$$

⁷The function $f_0(u)$ must not be confused with the bare coupling f_0 , which has no argument.

The finite volume version of the integral (2.52) at zero momentum can also be expressed in terms of f_0 :

$$\bar{b}(0) = -\frac{uf'_0(u)}{2M^2}. \quad (2.77)$$

We will need a second function defined similarly to $F(u)$:

$$H(u) = -\frac{1}{2} \int_0^\infty \frac{dt}{t^2} e^{-tu^2} \left(\sum_{k=1}^\infty e^{-\frac{k^2}{4t}} \right). \quad (2.78)$$

This function has the alternative representation

$$H(u) = \pi u + \frac{u^2}{2} \ln u - \frac{1}{4}(1 + 2\gamma)u^2 + 4\pi^2 G_1\left(\frac{u}{2\pi}\right) - \frac{\pi^2}{3}, \quad (2.79)$$

where

$$G_1(w) = \sum_{k=1}^\infty \left\{ \sqrt{k^2 + w^2} - k - \frac{w^2}{2k} \right\}. \quad (2.80)$$

The rest of the finite volume parameters appearing in the finite volume 2-point functions (2.56)-(2.65) can be given in terms of these functions:

$$\ell = \frac{uF'(u)}{4\pi} = \frac{1}{4\pi} + \frac{u}{2}f'_0(u), \quad h = -\frac{M^2}{2\pi}F(u) + \frac{M^2H(u)}{\pi u^2} \quad (2.81)$$

and

$$\bar{\rho} = \frac{3M^2}{16\pi} - M^2\ell - \frac{h}{2}. \quad (2.82)$$

For the cutoff dependence we will also need the functions $f_1(u)$ and $f_2(u)$ [8]:

$$f_1(u) = \frac{\pi}{6} \left[\frac{1}{12} - G_1\left(\frac{u}{2\pi}\right) \right] - \frac{u}{12} - \frac{u^2}{16\pi} \left[k + \frac{2}{3}G_0\left(\frac{u}{2\pi}\right) - \frac{1}{3} \right], \quad (2.83)$$

$$f_2(u) = 2\pi G_1\left(\frac{u}{2\pi}\right) - \frac{\pi}{6} + \frac{u}{2} + \frac{u^2}{8\pi}(2k - 1), \quad (2.84)$$

where $k = -\Gamma'(1) - \ln \pi + \frac{1}{2} \ln 2$.

Master equation for 2-point function in the strip geometry

Let us now apply Symanzik's method to calculate the lattice step scaling function by writing the master equation (1.2) for the 2-point function in finite volume:

$$\tilde{\mathcal{G}}_{\text{latt}}^{(2)} = y^2 \left\{ \bar{\mathcal{G}}_{(R)}^{(2)} + a^2 \sum_{i=1}^9 v_i \bar{\mathcal{G}}_{i(R)}^{(2)} + \mathcal{O}(a^4) \right\}. \quad (2.85)$$

From here the lattice correction to the finite volume mass gap is calculated as

$$M_{\text{latt}}^2 = M^2 - \frac{a^2}{f} \sum_{i=1}^9 v_i \bar{g}_i + \mathcal{O}(a^4), \quad (2.86)$$

where the coefficients \bar{g}_i are

$$\bar{g}_i = \bar{\mathcal{G}}_{i(R)}^{(2)}(0, iM). \quad (2.87)$$

If we take $\mu = 1/a$, i.e. identify the renormalization scale μ with the inverse lattice spacing, we can give the connection between the renormalized 't Hooft coupling f and the lattice parameters using (2.27):

$$\ln \xi = \frac{2\pi}{f} - \frac{\gamma}{2}, \quad (2.88)$$

where ξ is the lattice correlation length in infinite volume.

Using the results of the previous subsection, the coefficients in (2.86) are

$$\bar{g}_1 = \frac{M^2}{2\bar{b}(0)}, \quad (2.89)$$

$$\bar{g}_2 = \frac{f^2 h}{2} \left\{ \frac{\ell}{\bar{b}(0)} - M^2 \right\}, \quad (2.90)$$

$$\bar{g}_3 = \bar{g}_4 = \bar{g}_5 = \bar{g}_8 = 0, \quad (2.91)$$

$$\bar{g}_6 = 2fM^4 - \frac{2f\bar{\rho}}{\bar{b}(0)}, \quad (2.92)$$

$$\bar{g}_7 = -2\bar{g}_2, \quad (2.93)$$

$$\bar{g}_9 = -\frac{1}{\bar{b}(0)}. \quad (2.94)$$

3 Lattice actions

We consider here the large N limit of the standard, the constrained and the mixed lattice actions. Since the standard and the constrained actions can be obtained as special cases of the mixed action, we shall discuss in detail only the latter one.

3.1 Standard action

The action is given by

$$\mathcal{A}_{\text{st}}[\mathbf{S}] = \frac{\beta_{\text{st}}}{2} \sum_{x,\mu} (\partial_\mu \mathbf{S}_x)^2, \quad (3.1)$$

where $\mathbf{S}_x^2 = 1$ and ∂_μ denotes the forward lattice derivative.

As usual, the large N limit is taken as

$$\beta_{\text{st}} = \frac{N}{f_{\text{st}}}, \quad f_{\text{st}} = \text{fixed}. \quad (3.2)$$

Note for further reference that for large N the distribution of $(\partial_\mu \mathbf{S}_x)^2$ approaches a δ -function at $(\partial_\mu \mathbf{S}_x)^2 = f_{\text{st}}/2$ (for $D = 2$ Euclidean dimensions), as can be shown in perturbation theory.

For the standard action the infinite volume mass gap m is given by [12]

$$\frac{1}{f_{\text{st}}} = z \left[1 + \mathcal{O}(m^2 a^2) \right], \quad (3.3)$$

where z is given by eq. (1.19).

3.2 Constrained action

The action is the same as the standard one except that the configurations are restricted by the constraint $(\partial_\mu \mathbf{S}_x)^2 < \epsilon$:

$$\mathcal{A}_{\text{con}}[\mathbf{S}] = \begin{cases} \frac{1}{2}\beta_{\text{con}} \sum_{x,\mu} (\partial_\mu \mathbf{S}_x)^2 & \text{for } (\partial_\mu \mathbf{S}_x)^2 < \epsilon, \\ \infty & \text{otherwise,} \end{cases} \quad (3.4)$$

with $\beta_{\text{con}} = N/f_{\text{con}}$. As has been shown in [8], the constraint $(\partial_\mu \mathbf{S}_x)^2 < \epsilon$ is effectively replaced by the δ -function $\delta((\partial_\mu \mathbf{S}_x)^2 - \epsilon)$ in leading order in $1/N$. As a consequence, the physics for $\epsilon < f_{\text{con}}/2$ (to this order) is equivalent to the topological action [8] where only the constraint is present, i.e.

$$\mathcal{A}_{\text{top}}[\mathbf{S}] = \begin{cases} 0 & \text{for } (\partial_\mu \mathbf{S}_x)^2 < \epsilon, \\ \infty & \text{otherwise.} \end{cases} \quad (3.5)$$

The mass gap in this case is also given by (3.3) with f_{st} replaced by 2ϵ . (For $\epsilon > f_{\text{con}}/2$ the constraint is irrelevant in leading order, and the physics is given by the standard action, with $f_{\text{st}} = f_{\text{con}}$. We assume here that $\epsilon < f_{\text{con}}/2$.)

3.3 Mixed action

The mixed action [8] is given by

$$\mathcal{A}_{\text{mix}}[\mathbf{S}] = \frac{\beta_{\text{m}}}{2} \sum_{x,\mu} (\partial_\mu \mathbf{S}_x)^2 + \frac{\gamma_{\text{m}}}{4} \sum_{x,\mu} [(\partial_\mu \mathbf{S}_x)^2]^2. \quad (3.6)$$

The large N limit is taken as

$$\beta_{\text{m}} = \frac{N}{f_{\text{m}}}, \quad \gamma_{\text{m}} = \frac{2N}{\kappa_{\text{m}}^2}, \quad (3.7)$$

keeping f_{m} and κ_{m} constant.

The infinite volume mass gap is determined by the effective coupling \hat{f}_{m} defined by

$$\frac{1}{\hat{f}_{\text{m}}} = \frac{1}{2f_{\text{m}}} + \sqrt{\frac{1}{4f_{\text{m}}^2} + \frac{1}{\kappa_{\text{m}}^2}}, \quad (3.8)$$

and is given by the same expression as (3.3) with f_{st} replaced by \hat{f}_{m} .

Besides \hat{f}_{m} we introduce

$$r_{\text{m}} = \frac{\kappa_{\text{m}}}{f_{\text{m}}} \quad \text{and} \quad q_{\text{m}} = \frac{1}{2} \left(r_{\text{m}} + \sqrt{r_{\text{m}}^2 + 4} \right). \quad (3.9)$$

While the mass gap is determined by \hat{f}_{m} alone, the cutoff effects depend on the ratio r_{m} as well.

It is easy to see that $r_{\text{m}} = \infty$ ($q_{\text{m}} = \infty$) corresponds to the standard action. The choice $r_{\text{m}} = 0$ ($q_{\text{m}} = 1$) gives the “purely quartic” action (where $\beta_{\text{m}} = 0$). It is less obvious

that the $r_m \rightarrow -\infty$ ($q_m \rightarrow 0$) limit gives the constrained action. For large negative r_m the action density in (3.6) has a deep minimum at

$$(\partial_\mu \mathbf{S}_x)^2 = -\frac{\beta_m}{\gamma_m} = \frac{1}{2} \hat{f}_m \left(1 + \mathcal{O}(r_m^{-2}) \right), \quad (3.10)$$

which shows that this limit indeed corresponds to the constrained action with $\epsilon = \hat{f}_m/2$.

The technical details (introduction of auxiliary variables, the gap equation, the calculation of the 2-point and 4-point functions) can be found in [8], and we shall not repeat them here.

The cutoff dependence of the step scaling function $\Sigma(2, u, a/L)$ is described by (1.20), (1.23). The functions $T_i(u)$ are given by

$$T_i(u) = \frac{1}{f'_0(u'_\infty)} \left(\Phi_i(u) - \frac{1}{4} \Phi_i(u'_\infty) \right), \quad (3.11)$$

$$\begin{aligned} \Phi_0(u) &= f_1(u) + \frac{1}{8} u^2 f_0(u) - \frac{2f_2(u) - u^2 f_0(u)}{1 - \frac{2}{\pi} + q_m^2} \left(u f'_0(u) + \frac{1}{4\pi} \right), \\ \Phi_1(u) &= -\frac{1}{8} u^2, \\ \Phi_2(u) &= \frac{1}{1 + q_m^2} u^2. \end{aligned} \quad (3.12)$$

The functions $f_i(u)$ are defined in (2.74), (2.83), (2.84). Note that the coefficients $T_i(u)$ depend on the parameter q_m defined in (3.9). As shown above, the coefficients $t_i(u)$ and $\bar{t}_i(u)$, describing the cutoff dependence for the standard and constrained actions can be obtained from $T_i(u)$ by setting $q_m = \infty$ and $q_m = 0$, respectively. For the standard action ($q_m = \infty$) the z^2 term is absent.

4 Matching to the lattice results

As described in subsection 1.3, we calculate the 2-point and 4-point correlation functions on the lattice in infinite volume up to $\mathcal{O}(a^2)$, and compare these with the results obtained using Symanzik's effective action (1.1).

4.1 Matching the infinite volume 2-point function

In leading order in $1/N$ the 2-point function for the mixed action is the free lattice propagator

$$\tilde{\mathcal{G}}^{(2)}(k) = \frac{F}{\hat{k}^2 + m_0^2}, \quad (4.1)$$

where m_0 is expressed by the infinite-volume mass gap as

$$m_0 = \frac{2}{a} \sinh \frac{ma}{2}, \quad (4.2)$$

and F is related to the effective coupling \hat{f}_m by

$$\hat{f}_m = F \frac{r_m + \sqrt{r_m^2 + 4 - 4a^2 m_0^2 / F}}{r_m + \sqrt{r_m^2 + 4}} = F + \mathcal{O}(a^2 m_0^2) = F + \mathcal{O}(\exp(-4\pi/F)). \quad (4.3)$$

The gap equation is

$$\frac{1}{F} = \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{1}{\hat{q}^2 + m_0^2} = z(1 + \psi a^2 m_0^2) + \mathcal{O}(a^4 m_0^4), \quad (4.4)$$

where z is defined in (1.19) and

$$\psi = -\frac{1}{8} + \frac{1}{96\pi z}. \quad (4.5)$$

We now have

$$G_{\text{latt}}^{(2)}(k) = \frac{1}{z}(k^2 + m^2) + \frac{a^2}{z} \left\{ \frac{1}{12}(k^4 - m^4) - \psi m^2(k^2 + m^2) \right\} + \mathcal{O}(a^4). \quad (4.6)$$

According to Symanzik's conjecture this should be equal to

$$y^2 \left\{ G_R^{(2)}(k) + a^2 \sum_i v_i G_{i(R)}^{(2)}(k) + \mathcal{O}(a^4) \right\}. \quad (4.7)$$

Using (2.31) we find that the scaling parts match if the wave function renormalization constant is of the form

$$y^2 = \frac{1}{zf} (1 + w_8 a^2 m_0^2) + \mathcal{O}(a^4 m_0^4). \quad (4.8)$$

Matching the $\mathcal{O}(a^2)$ terms gives

$$\frac{1}{12}(k^4 - m^4) - \psi m^2(k^2 + m^2) = w_8 m^2(k^2 + m^2) + \frac{1}{f} \sum_i v_i G_{i(R)}^{(2)}(k). \quad (4.9)$$

Using the formulas (2.32)-(2.40) we find

$$v_6 = \frac{1}{24}, \quad (4.10)$$

$$\frac{v_3}{2} + v_5 = \frac{1}{16}, \quad (4.11)$$

$$v_5 + \frac{v_8}{2m^2} = -\frac{1}{16} - \psi - w_8, \quad (4.12)$$

$$v_1 - \frac{2v_9}{m^2} = -\frac{f}{96\pi}. \quad (4.13)$$

4.2 Matching the infinite volume 4-point function (standard action)

We analyse first the ST case. In this case the 4-point function is

$$\tilde{\mathcal{G}}_{\text{latt}}^{(4)} = -\frac{2f_{\text{st}}^2}{NB_{\text{latt}}(p)} \frac{1}{\hat{p}_1^2 + m_0^2} \frac{1}{\hat{p}_2^2 + m_0^2} \frac{1}{\hat{q}_1^2 + m_0^2} \frac{1}{\hat{q}_2^2 + m_0^2}, \quad (4.14)$$

where $p = p_1 + p_2 = -q_1 - q_2$ and

$$B_{\text{latt}}(p) = \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{1}{(\hat{q}^2 + m_0^2) [m_0^2 + (\widehat{p+q})^2]}. \quad (4.15)$$

Expanding in the lattice spacing we have

$$B_{\text{latt}}(p) = b(p) + a^2 b_1(p) + \mathcal{O}(a^4) . \quad (4.16)$$

The functions $b(p)$ and $b_1(p)$ are given in appendix C.

The amputated 4-point function on the lattice is then

$$G_{\text{latt}}^{(4)} = -\frac{2}{N z^2 b(p)} + \frac{2a^2}{N z^2 b(p)} \left\{ 2\psi m^2 + \frac{b_1(p)}{b(p)} - \frac{1}{12} \left[\frac{p_1^4 - m^4}{p_1^2 + m^2} + \frac{p_2^4 - m^4}{p_2^2 + m^2} + \frac{q_1^4 - m^4}{q_1^2 + m^2} + \frac{q_2^4 - m^4}{q_2^2 + m^2} \right] \right\} + \mathcal{O}(a^4) , \quad (4.17)$$

which we have to match to the DR result,

$$\frac{1}{z^2 f^2} (1 + 2w_8 a^2 m_0^2 + \mathcal{O}(a^4 m_0^4)) \left\{ G_R^{(4)} + a^2 \sum_i v_i G_{i(R)}^{(4)} + \mathcal{O}(a^4) \right\} . \quad (4.18)$$

Using (2.42) we see that the scaling pieces indeed agree and from the rest we have

$$2\psi m^2 + \frac{b_1(p)}{b(p)} - \frac{1}{12} \left[\frac{p_1^4 - m^4}{p_1^2 + m^2} + \frac{p_2^4 - m^4}{p_2^2 + m^2} + \frac{q_1^4 - m^4}{q_1^2 + m^2} + \frac{q_2^4 - m^4}{q_2^2 + m^2} \right] = -2w_8 m^2 + \frac{N b(p)}{2f^2} \sum_i v_i G_{i(R)}^{(4)} . \quad (4.19)$$

Here we have to use the results (2.43)-(2.51). Because contributions of some operators have momentum dependence not occurring in the above expression we first have

$$v_2 = v_4 = v_7 = 0 . \quad (4.20)$$

Next we get

$$v_3 = \frac{1}{4} , \quad (4.21)$$

which combined with the results of the previous subsection gives

$$v_5 = -\frac{1}{16} \quad \text{and} \quad v_8 = -2m^2(\psi + w_8) . \quad (4.22)$$

The last matching equation is

$$\frac{v_1}{2f} = b_1(p) - \frac{m^4}{12} \frac{\partial b(p)}{\partial m^2} - \frac{1}{6} Y . \quad (4.23)$$

After inserting the expressions for $b(p)$ and $b_1(p)$ given in appendix C, we obtain the simple result

$$v_1 = \frac{f}{4} \left(z - \frac{1}{8\pi} \right) = \frac{1}{4} + \frac{f}{16\pi} \left(\ln \frac{8}{\pi} - \frac{1}{2} - \Gamma'(1) \right) . \quad (4.24)$$

Eqs. (4.10), (4.20), (4.24) are consistent with (3.30) from [7] where we found to first order in coupling constant PT⁸:

$$v_1 = \frac{1}{4} + \mathcal{O}(f) , \quad v_2 = \mathcal{O}(f) , \quad v_3 = \frac{1}{4} + \mathcal{O}(f) , \quad v_4 = \mathcal{O}(f) , \\ v_5 = -\frac{1}{16} + \mathcal{O}(f) , \quad v_6 = \frac{1}{24} + \mathcal{O}(f) , \quad v_7 = \mathcal{O}(f) . \quad (4.25)$$

⁸Note the extra N factor in our U_1 with respect to U_1 of that paper.

4.3 Matching the infinite volume 4-point function (mixed action)

Referring to appendix B of [8] the mixed case amounts to adding an extra piece to the propagator of the auxiliary field,

$$\tilde{\Delta}(p) \rightarrow \tilde{\Delta}(p) + a^2 \sum_{\mu,\nu} X_\mu(p_1, p_2) X_\nu(q_1, q_2) \tilde{\Delta}_{\mu\nu}(p) \quad (4.26)$$

where

$$\tilde{\Delta}(p) = \frac{1}{B_{\text{latt}}(p)}, \quad (4.27)$$

$$\tilde{\Delta}_{\mu\nu}(p) \sim \frac{2\pi}{F^2} \left[k_1 \left(\delta_{\mu\nu} - \frac{1}{2} \right) + k_2 \right] + \mathcal{O}(a^2), \quad (4.28)$$

$$k_1 = \frac{1}{\pi - 2 + \pi q_m^2}, \quad k_2 = \frac{1}{2\pi(1 + q_m^2)}, \quad (4.29)$$

and

$$X_\mu(p_1, p_2) = -\hat{p}_{1\mu} \hat{p}_{2\mu} - \frac{h_\mu(p)}{h(p)}. \quad (4.30)$$

Here $h(p) = B_{\text{latt}}(p)$ and $h_\mu(p)$ are given in (4.15), (C.31).

Expressing the terms with specific momentum dependence by the appropriate combinations of $G_{i(R)}^{(4)}$, after some algebra we get for the additional part of the 4-point function of the mixed action

$$\begin{aligned} \sum_{\mu,\nu} X_\mu(p_1, p_2) X_\nu(q_1, q_2) \tilde{\Delta}_{\mu\nu}(p) &= \frac{2\pi N}{F^2} \left\{ \frac{k_1}{8f^3} \left(G_{7(R)}^{(4)} - 2G_{2(R)}^{(4)} \right) \right. \\ &\quad \left. + k_2 \left(\frac{z^2}{f} \left[G_{1(R)}^{(4)} + \frac{m^2}{2} G_{9(R)}^{(4)} \right] + \frac{z}{f^2} G_{3(R)}^{(4)} + \frac{1}{f^3} G_{4(R)}^{(4)} + \frac{zm^2}{f^2} G_{8(R)}^{(4)} \right) \right\} + \mathcal{O}(a^2). \end{aligned} \quad (4.31)$$

The matching gives

$$v_i^{\text{mix}} = v_i^{\text{st}} + x_i, \quad i \neq 5, \quad (4.32)$$

where the x_i are determined through

$$\sum_{i \neq 5} x_i G_{i(R)}^{(4)} = -\frac{2f^2 F^2}{N} \sum_{\mu,\nu} X_\mu(p_1, p_2) X_\nu(q_1, q_2) \tilde{\Delta}_{\mu\nu}(p) + \mathcal{O}(a^2), \quad (4.33)$$

and so we get

$$v_1^{\text{mix}} = f \left[-4\pi z^2 k_2 + \frac{z}{4} - \frac{1}{32\pi} \right], \quad (4.34)$$

$$v_2^{\text{mix}} = \frac{\pi k_1}{f}, \quad (4.35)$$

$$v_3^{\text{mix}} = \frac{1}{4} - 4\pi z k_2, \quad (4.36)$$

$$v_4^{\text{mix}} = -\frac{4\pi k_2}{f}, \quad (4.37)$$

$$v_5^{\text{mix}} = -\frac{1}{16} + 2\pi z k_2, \quad (4.38)$$

$$v_6^{\text{mix}} = \frac{1}{24}, \quad (4.39)$$

$$v_7^{\text{mix}} = -\frac{\pi k_1}{2f}, \quad (4.40)$$

$$v_8^{\text{mix}} = -2m^2 [\psi + 2\pi z k_2 + w_8], \quad (4.41)$$

$$v_9^{\text{mix}} = \frac{m^2}{2} \left[\frac{f}{96\pi} + v_1^{\text{mix}} \right]. \quad (4.42)$$

The coefficient v_8^{mix} can be determined only up to the parameter w_8 which can be fixed if we first impose a renormalization condition on the lattice 2-point functions.

Note that the matching coefficients for the mixed action are non-perturbative in the coupling constant, some of them contain a $1/f$ factor. This explains why the leading artifact in this case is $a^2 \ln^2 a$ as opposed to $a^2 \ln a$ for the standard action. The mixed action is perturbative (i.e. the quadratic part dominates) only when $f_m \ll \kappa_m$, i.e. $q_m \gg 1$. In this case the q_m^2 in the denominator of (4.29) compensates the $1/f$ factor in the continuum limit.

To minimize the lattice artifacts for the step scaling function one has to take [8] $q_m^2 = 8/\hat{f}_m + \mathcal{O}(1)$. Inserting this and $z = 1/f + \mathcal{O}(1)$, and $1/\hat{f}_m = 1/f + \mathcal{O}(1)$ into (4.34)-(4.42) the coefficients v_1 and v_3 become $\mathcal{O}(f)$ while for the standard action they are $\mathcal{O}(1)$. As has been shown in [7], indeed the operator U_1 is responsible for the leading lattice artifacts.

4.4 The matching for the step scaling function

Let us parametrize the “couplings” \bar{g}_i defined in (2.89)-(2.94) as

$$\bar{g}_i = 2f^{e_i} M^4(L) \frac{\Psi_i(u)}{u^3 f'_0(u)}. \quad (4.43)$$

Using also the finite volume results from appendix B of [8], we have

$$e_1 = 0, \quad \Psi_1(u) = -\frac{u^2}{2}, \quad (4.44)$$

$$e_2 = 2, \quad \Psi_2(u) = -\frac{1}{2} \left(\frac{1}{4\pi} + u f'_0(u) \right) (2f_2(u) - u^2 f_0(u)), \quad (4.45)$$

$$e_6 = 1, \quad \Psi_6(u) = 24f_1(u) + 3u^2 f_0(u) - \frac{3u^2}{8\pi}, \quad (4.46)$$

$$e_7 = 2, \quad \Psi_7(u) = -2\Psi_2(u), \quad (4.47)$$

$$e_9 = 0, \quad \Psi_9(u) = L^2, \quad (4.48)$$

and

$$\Psi_3(u) = \Psi_4(u) = \Psi_5(u) = \Psi_8(u) = 0. \quad (4.49)$$

It is interesting to observe that the off-shell operators U_3 , U_4 , U_5 and U_8 do not contribute to the finite volume mass. U_9 does contribute, but since $\Psi_9(u'_\infty) = (2L)^2$, its contribution to the step scaling function also vanishes:

$$\Psi_9(u) - \frac{1}{4}\Psi_9(u'_\infty) = 0. \quad (4.50)$$

Using the mass formula (2.86) we find

$$\nu(u, z) = \frac{1}{f'_0(u'_\infty)} \left\{ \Psi(u) - \frac{1}{4}\Psi(u'_\infty) \right\} \quad (4.51)$$

with

$$\Psi(u) = \frac{v_1}{f}\Psi_1(u) + f(v_2 - 2v_7)\Psi_2(u) + v_6\Psi_6(u). \quad (4.52)$$

Inserting the coefficients v_i given by eqs. (4.34)-(4.42) one obtains for the mixed action⁹

$$v_1 = f \left(-\frac{2z^2}{1 + q_m^2} + \frac{z}{4} - \frac{1}{32\pi} \right), \quad (4.53)$$

$$v_2 - 2v_7 = \frac{2\pi}{\pi - 2 + \pi q_m^2} \frac{1}{f}, \quad (4.54)$$

$$v_6 = \frac{1}{24}. \quad (4.55)$$

These results agree with those of [8]. Note that matching the finite volume mass gap does not fix completely the coefficients v_i – it yields only the relations given above. Also note that the coefficients of the effective action obtained by matching in finite volume do not depend on the volume, as expected by general considerations [13, 14].

⁹The results for the standard and constrained actions can be obtained from here by setting q_m^2 to ∞ and 0, respectively.

5 Conclusions

The extrapolation of lattice data to the continuum limit is a crucial step in the determination of low energy physical quantities in Quantum Chromodynamics. Our present theoretical understanding of lattice artifacts is based on considerations of Symanzik. He argued that asymptotically the leading ultraviolet cutoff effects are described by an effective continuum Lagrangian containing a finite number of higher dimensional operators, restricted by symmetries only of the underlying lattice regularization, and with coefficients depending on the lattice spacing.

In this paper we have given additional support for Symanzik’s theory of lattice artifacts, by determining the local effective Lagrangian for a class of lattice actions of the 2-dimensional nonlinear $O(N)$ sigma model in a non-perturbative setting of the $1/N$ -expansion at the leading order. The class of models considered includes also ones which are considered to belong to the same universality class as standard actions, but for which the usual perturbative expansions are (as yet) not available.

The effective actions are shown to reproduce previously computed lattice artifacts of the associated step scaling functions which are defined in finite volume. Once established the effective action can be used to predict cutoff effects for various observables.

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A Two-point functions with operator insertion in finite volume using dimensional regularization

Ordinary 2-point function and source-dependent operators

From the leading order large N Feynman rules we see that the sigma field propagator is of the free form

$$\overline{G}^{(2)} = f_0(k^2 + M^2). \quad (\text{A.1})$$

We see that the wave function renormalization constant Z is the same as appearing in the coupling renormalization and the renormalized 2-point function is

$$\overline{G}_{(R)}^{(2)} = f(k^2 + M^2). \quad (\text{A.2})$$

The correlators of the source-dependent operators U_3 , U_4 and U_5 are given in [7]. The correlators of U_3 are simply related to the ordinary correlation functions (without operator

insertion):

$$\tilde{\mathcal{G}}_3^{(r)}(p_1, \dots, p_r) = \frac{1}{4} \left(\sum_{k=1}^r p_k^2 \right) \tilde{\mathcal{G}}^{(r)}(p_1, \dots, p_r). \quad (\text{A.3})$$

From this representation it is clear that we have for the operator renormalization constant $Z_3 = 1$.

The correlators of U_4 are related [7] to those of the isospin tensor operator $S^a S^b - \frac{1}{N} \delta^{ab}$. Its 2-point correlation function vanishes and therefore the operator renormalization constant Z_4 can only be calculated from its 4-point function.

The operator U_5 corresponds to a contact term: its 2-point function is given by

$$\overline{G}_5^{(2)} = f_0(k^2 + M^2)^2 \quad (\text{A.4})$$

and all higher correlation functions vanish: $\overline{G}_5^{(r)} = 0$ for $r > 2$. We see that $Z_5 = 1$ and the renormalized 2-point function is

$$\overline{G}_{5(R)}^{(2)} = f(k^2 + M^2)^2. \quad (\text{A.5})$$

Finally for the source-dependent operator U_8 we have

$$\tilde{\mathcal{G}}_8^{(r)} = \frac{r}{4} \tilde{\mathcal{G}}^{(r)} \quad (\text{A.6})$$

for all r . From the $r = 2$ case we obtain

$$\overline{G}_8^{(2)} = \frac{f_0}{2}(k^2 + M^2), \quad Z_8 = 1, \quad \overline{G}_{8(R)}^{(2)} = \frac{f}{2}(k^2 + M^2). \quad (\text{A.7})$$

The coupling-related operator U_9

This operator is related to the integral of the action density

$$\frac{1}{2g_0^2} \int d^D x \partial_\mu S \cdot \partial_\mu S \quad (\text{A.8})$$

and if we denote the corresponding correlation functions by $\mathcal{G}_{\mathcal{A}}^X$, where X is any combination of operators, we can derive the following Ward identity:

$$\mathcal{G}_{\mathcal{A}}^X = g_0^2 \frac{\partial}{\partial g_0^2} \mathcal{G}^X = f_0 \frac{\partial}{\partial f_0} \mathcal{G}^X. \quad (\text{A.9})$$

Applying this to the 2-point function we get

$$\tilde{\mathcal{G}}_{\mathcal{A}}^{(2)} = \frac{f_0}{k^2 + M^2} - \frac{1}{\overline{B}(0)(k^2 + M^2)^2}, \quad (\text{A.10})$$

where we used the relation

$$f_0 \frac{\partial M^2}{\partial f_0} = \frac{1}{f_0 \overline{B}(0)}, \quad (\text{A.11})$$

which can be obtained from the gap equation (2.69). Here $\overline{B}(0)$ denotes the finite volume version of the integral (2.4) evaluated at $p = 0$:

$$\overline{B}(0) = \int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + M^2)^2}. \quad (\text{A.12})$$

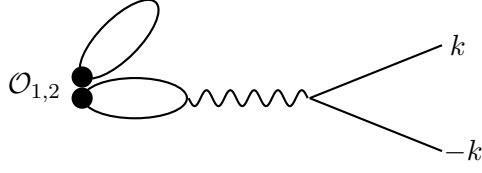


Figure 1.

A contribution of the 2-point function with insertion of the operator \mathcal{O}_1 or \mathcal{O}_2 . The two black blobs denote the two sigma-field contractions in (1.5), and have similar meaning in all other figures. The isospin indices of the outgoing legs are contracted.

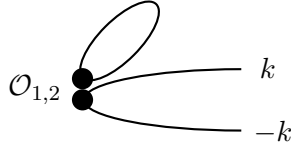


Figure 2.

Using the definition (2.12) we find

$$\overline{G}_9^{(2)} = -\frac{1}{\overline{B}(0)}. \quad (\text{A.13})$$

From this we see that $Z_9 = Z$ and

$$\overline{G}_{9(R)}^{(2)} = -\frac{1}{\overline{b}(0)}. \quad (\text{A.14})$$

The operators U_1, U_2

Let us introduce the index notation $\dot{1}, \dot{2}$ for correlation functions with insertion of the space integral of the local operators

$$\frac{1}{g_0^2} \mathcal{O}_i, \quad i = 1, 2. \quad (\text{A.15})$$

We then have for the 2-point functions (see figures 1, 2):

$$\overline{G}_1^{(2)} = \frac{f_0^2}{2} \left(\sum_{\mu=1}^D \mathcal{H}_\mu \right) \left(k^2 - \frac{1}{\overline{B}(0)} \sum_{\nu=1}^D \mathcal{L}_\nu \right), \quad (\text{A.16})$$

$$\overline{G}_2^{(2)} = \frac{f_0^2}{2} \sum_{\mu=1}^D \mathcal{H}_\mu \left(k_\mu^2 - \frac{\mathcal{L}_\mu}{\overline{B}(0)} \right), \quad (\text{A.17})$$

where

$$\mathcal{H}_\mu = \int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^2}{q^2 + M^2}, \quad \mathcal{L}_\mu = \int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^2}{(q^2 + M^2)^2}. \quad (\text{A.18})$$

These integrals can be expressed in terms of the differences

$$\mathcal{H}_1 - \mathcal{H}_2 = h, \quad \mathcal{L}_1 - \mathcal{L}_2 = \ell \quad (\text{A.19})$$

and sums

$$\sum_{\mu=1}^D \mathcal{H}_\mu = -\frac{M^2}{f_0}, \quad \sum_{\mu=1}^D \mathcal{L}_\mu = \frac{1}{f_0} - M^2 \overline{B}(0). \quad (\text{A.20})$$

Taking into account that there is just one momentum component (q_1) with discrete spectrum and $D - 1$ continuous ones, we have

$$\sum_{\mu=1}^D \mathcal{H}_\mu = D\mathcal{H}_2 + h \quad (\text{A.21})$$

and similarly for \mathcal{L}_μ .

Using the above relations we can now write

$$\overline{G}_1^{(2)} = -\frac{M^2 f_0}{2} \left(k^2 - \frac{1}{\overline{B}(0)} \sum_{\nu=1}^D \mathcal{L}_\nu \right) = -\frac{M^2 f_0}{2} (k^2 + M^2) + \frac{M^2}{2\overline{B}(0)} \quad (\text{A.22})$$

leading to (using (2.13))

$$\overline{G}_1^{(2)} = \frac{M^2}{2\overline{B}(0)}, \quad Z_1 = Z, \quad \overline{G}_{1(R)}^{(2)} = \frac{M^2}{2\overline{b}(0)}. \quad (\text{A.23})$$

An alternative way to express $\overline{G}_1^{(2)}$ is

$$\overline{G}_1^{(2)} = \frac{f_0^2}{2} \left\{ D\mathcal{H}_2 k^2 + h k^2 - \frac{D^2 \mathcal{H}_2 \mathcal{L}_2}{\overline{B}(0)} - \frac{D(h\mathcal{L}_2 + \ell\mathcal{H}_2)}{\overline{B}(0)} - \frac{\ell h}{\overline{B}(0)} \right\}. \quad (\text{A.24})$$

Comparing to

$$\overline{G}_2^{(2)} = \frac{f_0^2}{2} \left\{ \mathcal{H}_2 k^2 + h k_1^2 - \frac{D\mathcal{H}_2 \mathcal{L}_2}{\overline{B}(0)} - \frac{h\mathcal{L}_2 + \ell\mathcal{H}_2}{\overline{B}(0)} - \frac{\ell h}{\overline{B}(0)} \right\} \quad (\text{A.25})$$

we obtain using the definition in (2.13)

$$\overline{G}_2^{(2)} = \frac{f_0^2 h}{2} \left\{ k^2 - D k_1^2 + (D - 1) \frac{\ell}{\overline{B}(0)} \right\}. \quad (\text{A.26})$$

Here we can demonstrate that our results are independent of the way we regularize the finite volume problem. Indeed, if, instead of (A.21) we write

$$\sum_{\mu=1}^D \mathcal{H}_\mu = D\mathcal{H}_2 + (D - 1)h \quad (\text{A.27})$$



Figure 3.

corresponding to $D-1$ discrete and one continuous momentum components, we find instead of (A.26)

$$\overline{G}_2^{(2)} = \frac{f_0^2 h}{2} \left\{ Dk_2^2 - k^2 + (D-1) \frac{\ell}{\overline{B}(0)} \right\}. \quad (\text{A.28})$$

In the $D \rightarrow 2$ limit, both (A.26) and (A.28) lead to

$$Z_2 = \frac{1}{Z}, \quad \overline{G}_{2(R)}^{(2)} = \frac{f^2 h}{2} \left\{ k_2^2 - k_1^2 + \frac{\ell}{\overline{b}(0)} \right\}. \quad (\text{A.29})$$

The integrals (2.66) corresponding to the differences h and ℓ are finite.

The tensor operators U_6, U_7

The main part of U_7 is the term

$$\frac{N}{f_0} \sum_{\mu=1}^D (\partial_\mu S \cdot \partial_\mu S)^2 \quad (\text{A.30})$$

and corresponds to the 2-point function

$$4f_0^2 \sum_{\mu=1}^D \mathcal{H}_\mu \left(k_\mu^2 - \frac{\mathcal{L}_\mu}{\overline{B}(0)} \right) = 8\overline{G}_2^{(2)}. \quad (\text{A.31})$$

Combining this with the remaining contributions from (2.10) we find

$$\overline{G}_7^{(2)} = \left(8 - \frac{16}{D+2} \right) \overline{G}_2^{(2)} - \frac{8}{D+2} \overline{G}_1^{(2)} = -\frac{8}{D+2} \overline{G}_2^{(2)} \quad (\text{A.32})$$

and hence

$$Z_7 = \frac{1}{Z}, \quad \overline{G}_{7(R)}^{(2)} = -2\overline{G}_{2(R)}^{(2)} = f^2 h \left\{ k_1^2 - k_2^2 - \frac{\ell}{\overline{b}(0)} \right\}. \quad (\text{A.33})$$

The main part of U_6 is

$$Q_6 = \frac{N}{f_0} \sum_{\mu=1}^D S \cdot \partial_\mu^4 S \quad (\text{A.34})$$

and corresponds to the 2-point function (see figures 3,4)

$$2f_0 \left(\sum_{\mu=1}^D k_\mu^4 - \frac{1}{\overline{B}(0)} \sum_{\mu=1}^D \mathcal{K}_\mu \right), \quad (\text{A.35})$$

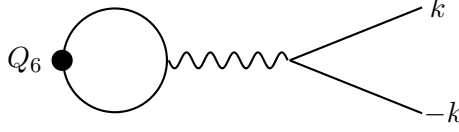


Figure 4.

where

$$\mathcal{K}_\mu = \int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + M^2)^2}. \quad (\text{A.36})$$

Using the definition (2.9) the 2-point function of U_6 becomes

$$\overline{G}_6^{(2)} = 2f_0 \sum_{\mu=1}^D k_\mu^4 + \frac{6f_0}{D+2} (M^4 - (k^2)^2) - \frac{2f_0}{\overline{B}(0)} \overline{\mathcal{R}}, \quad (\text{A.37})$$

with

$$\overline{\mathcal{R}} = \sum_{\mu=1}^D \mathcal{K}_\mu + \frac{6M^2}{D+2} \frac{1}{f_0} = \overline{\rho} + \mathcal{O}(\varepsilon). \quad (\text{A.38})$$

The final result for this operator is

$$Z_6 = 1, \quad \overline{G}_{6(R)}^{(2)} = 2fk^4 + \frac{3f}{2} (M^4 - (k^2)^2) - \frac{2f\overline{\rho}}{\overline{b}(0)}. \quad (\text{A.39})$$

To obtain the infinite volume results (2.31)-(2.40) from the above formulae is straightforward. One has to use the $L \rightarrow \infty$ limits

$$b(0) = \frac{1}{4\pi m^2} \quad \text{and} \quad \rho = \frac{3m^2}{16\pi}. \quad (\text{A.40})$$

Calculation of the finite volume integrals

By introducing a Feynman parameter, we can rewrite the gap equation

$$\frac{1}{f_0} = \int_0^\infty dt e^{-tM^2} \int_{(L)} \frac{d^D q}{(2\pi)^D} e^{-tq^2} \quad (\text{A.41})$$

and similarly the basic loop integral at zero momentum:

$$\overline{B}(0) = \int_0^\infty t dt e^{-tM^2} \int_{(L)} \frac{d^D q}{(2\pi)^D} e^{-tq^2}. \quad (\text{A.42})$$

Let us concentrate on the contribution of a single one dimensional integral in the last D -dimensional integral:

$$\int_{-\infty}^\infty \frac{dq}{2\pi} e^{-tq^2} = \frac{1}{\sqrt{4\pi t}}. \quad (\text{A.43})$$

The corresponding “integral”, if the momentum variable is discrete, is really an infinite sum of the form

$$\frac{1}{L} \sum_{n=-\infty}^\infty e^{-t(\frac{2\pi n}{L})^2}. \quad (\text{A.44})$$

Here and in all other similar sums we encounter the theta-function $S(x)$ defined by

$$S(x) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2}. \quad (\text{A.45})$$

Using this notation, (A.44) can be written as

$$\frac{1}{L} S\left(\frac{4\pi t}{L^2}\right). \quad (\text{A.46})$$

A useful identity satisfied by the function $S(x)$ is

$$S(x) = \frac{1}{\sqrt{x}} S\left(\frac{1}{x}\right), \quad (\text{A.47})$$

which can be proven using the Poisson resummation formula. Thus an alternative form of (A.44) is

$$\frac{1}{\sqrt{4\pi t}} S\left(\frac{L^2}{4\pi t}\right). \quad (\text{A.48})$$

Now we rewrite the gap equation and the zero momentum loop integral as

$$\frac{1}{f_0} = \int_0^\infty dt e^{-tM^2} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right) = \int_0^\infty dt e^{-tm^2} \frac{1}{(4\pi t)^{D/2}}, \quad (\text{A.49})$$

$$\overline{B}(0) = \int_0^\infty t dt e^{-tM^2} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right). \quad (\text{A.50})$$

The advantage of using the infinite sum $S(x)$ with this argument is that the $n = 0$ term is identical with the infinite volume limit of the given quantity, while the $n \neq 0$ terms are UV finite [14].

From (A.49) we obtain the equation

$$\int_0^\infty dt \left(e^{-tM^2} - e^{-tm^2} \right) \frac{1}{(4\pi t)^{D/2}} + \int_0^\infty dt e^{-tM^2} \frac{1}{(4\pi t)^{D/2}} \left[S\left(\frac{L^2}{4\pi t}\right) - 1 \right] = 0. \quad (\text{A.51})$$

Both integrals have finite $D \rightarrow 2$ limits and evaluating the first one and using the definition (2.71) in the second one leads to the final result (2.70) for calculating the LWW coupling $u = LM$. The expression (A.50) is also finite for $D \rightarrow 2$ and gives

$$4\pi M^2 \overline{b}(0) = u^2 \int_0^\infty dt e^{-tu^2} \left\{ 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{k^2}{4t}} \right\} = 1 - uF'(u). \quad (\text{A.52})$$

Next we consider integrals of the form

$$\int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^2}{(q^2 + M^2)^{\sigma+1}} = \frac{1}{\Gamma(\sigma+1)} \int_0^\infty t^\sigma dt e^{-tM^2} \int_{(L)} \frac{d^D q}{(2\pi)^D} q_\mu^2 e^{-tq^2}. \quad (\text{A.53})$$

We will need integrals with $\sigma = 0, 1$. If $\mu \neq 1$ then the last integral is simply

$$\frac{1}{2t} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right) \quad (\text{A.54})$$

whereas for $\mu = 1$ it is

$$\frac{1}{2t} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right) - \frac{1}{(4\pi t)^{D/2}} \frac{\partial}{\partial t} S\left(\frac{L^2}{4\pi t}\right). \quad (\text{A.55})$$

To evaluate h and ℓ defined in (2.66) we only need to integrate the difference between the above two terms, which is finite as $D \rightarrow 2$:

$$\begin{aligned} & - \int_0^\infty t^\sigma dt e^{-tM^2} \frac{1}{4\pi t} \frac{\partial}{\partial t} \left[S\left(\frac{L^2}{4\pi t}\right) - 1 \right] \\ & = - \frac{(L^2)^{\sigma-1}}{2\pi} \int_0^\infty dt e^{-tu^2} \{ t^{\sigma-1} u^2 + (1-\sigma)t^{\sigma-2} \} \sum_{k=1}^\infty e^{-\frac{k^2}{4t}}. \end{aligned} \quad (\text{A.56})$$

For $\sigma = 1$ we obtain from this the first and for $\sigma = 0$ the second relation in (2.81).

Finally we consider

$$\int_{(L)} \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + M^2)^2} = \int_0^\infty t dt e^{-tM^2} \int_{(L)} \frac{d^D q}{(2\pi)^D} q_\mu^4 e^{-tq^2}. \quad (\text{A.57})$$

Here for $\mu \neq 1$ the last integral is

$$\frac{3}{4t^2} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right) \quad (\text{A.58})$$

whereas for $\mu = 1$ it is

$$\frac{1}{(4\pi t)^{D/2}} \left\{ \frac{3}{4t^2} S\left(\frac{L^2}{4\pi t}\right) - \frac{1}{t} \frac{\partial}{\partial t} S\left(\frac{L^2}{4\pi t}\right) + \frac{\partial^2}{\partial t^2} S\left(\frac{L^2}{4\pi t}\right) \right\}. \quad (\text{A.59})$$

Summing over μ , as required by the first term in (2.68), we get

$$\int_0^\infty t dt e^{-tM^2} \frac{1}{(4\pi t)^{D/2}} \left\{ \frac{3D}{4t^2} S\left(\frac{L^2}{4\pi t}\right) - \frac{1}{t} \frac{\partial}{\partial t} S\left(\frac{L^2}{4\pi t}\right) + \frac{\partial^2}{\partial t^2} S\left(\frac{L^2}{4\pi t}\right) \right\}. \quad (\text{A.60})$$

This has to be combined with the second term in (2.68), which is of the form

$$\frac{6M^2}{D+2} \int_0^\infty dt e^{-tM^2} \frac{1}{(4\pi t)^{D/2}} S\left(\frac{L^2}{4\pi t}\right). \quad (\text{A.61})$$

Let us now separate the contributions into a “finite” part, which is obtained by the replacement

$$S\left(\frac{L^2}{4\pi t}\right) \longrightarrow S\left(\frac{L^2}{4\pi t}\right) - 1 \quad (\text{A.62})$$

corresponding to UV finite integrals and an “infinite” part, which corresponds to the substitution

$$S\left(\frac{L^2}{4\pi t}\right) \longrightarrow 1. \quad (\text{A.63})$$

We first evaluate the “infinite” contributions. We find

$$\begin{aligned} & \frac{3D}{4(4\pi)^{D/2}} \int_0^\infty dt e^{-tM^2} t^{-1-D/2} + \frac{6M^2}{(D+2)(4\pi)^{D/2}} \int_0^\infty dt e^{-tM^2} t^{-D/2} \\ & = \frac{3M^D}{(4\pi)^{D/2}} \left\{ \frac{D}{4} \Gamma(-D/2) + \frac{2}{D+2} \Gamma(1-D/2) \right\} = \frac{3M^D}{(D+2)(4\pi)^{D/2}} \Gamma(2-D/2). \end{aligned} \quad (\text{A.64})$$

In the $D \rightarrow 2$ limit this is finite and gives $3M^2/16\pi$. The “finite” part gives

$$\begin{aligned}
& \int_0^\infty t \, dt \, e^{-tM^2} \frac{1}{4\pi t} \left\{ \frac{3}{2t^2} \left[S\left(\frac{L^2}{4\pi t}\right) - 1 \right] - \frac{1}{t} \frac{\partial}{\partial t} S\left(\frac{L^2}{4\pi t}\right) + \frac{\partial^2}{\partial t^2} S\left(\frac{L^2}{4\pi t}\right) \right\} \\
& + \frac{3M^2}{2} \int_0^\infty dt \, e^{-tM^2} \frac{1}{4\pi t} \left[S\left(\frac{L^2}{4\pi t}\right) - 1 \right] \\
& = \frac{M^2}{2\pi} \int_0^\infty dt \, e^{-tu^2} \left(\sum_{k=1}^\infty e^{-\frac{k^2}{4t}} \right) \left\{ \frac{1}{2u^2 t^2} + \frac{1}{2t} + u^2 \right\} \\
& = -\frac{M^2}{2\pi u^2} H(u) + \frac{M^2}{4\pi} F(u) - \frac{M^2}{4\pi} u F'(u) = -M^2 \ell - \frac{h}{2}.
\end{aligned} \tag{A.65}$$

Adding the two contributions we finally have the result (2.82).

B Four-point functions with operator insertion in infinite volume using dimensional regularization

Ordinary 4-point function and source-dependent operators

The leading large N 4-point functions are of order $1/N$. Using the Feynman rules given in section 2, the leading order ordinary 4-point function is

$$G^{(4)} = -\frac{2f_0^2}{NB(p)}. \tag{B.1}$$

After renormalization we get for the renormalized 4-point function (2.42). The 4-point function of the source-dependent operator U_3 can be obtained from the above by multiplying it by a momentum-dependent factor:

$$G_3^{(4)} = -\frac{f_0^2}{2NB(p)}(p_1^2 + p_2^2 + q_1^2 + q_2^2) \tag{B.2}$$

and after renormalization this becomes (2.45). Correlation functions of the source-dependent operator U_4 can be calculated using the rules given in [7]. For the 4-point function we get

$$G_4^{(4)} = \frac{f_0^3}{N} \left(\frac{p_1^2 + p_2^2}{2} + m^2 \right) \left(\frac{q_1^2 + q_2^2}{2} + m^2 \right). \tag{B.3}$$

This allows us to determine Z_4 (the only operator renormalization constant not determined by the 2-point functions). We obtain $Z_4 = 1/Z$ and (2.46). Next we recall that

$$G_5^{(4)} = G_{5(R)}^{(4)} = 0, \tag{B.4}$$

since all $r > 2$ correlation functions of U_5 vanish. Finally the 4-point correlators of U_8 coincide with the ordinary ones:

$$G_8^{(4)} = -\frac{2f_0^2}{NB(p)}, \quad G_{8(R)}^{(4)} = -\frac{2f^2}{Nb(p)}. \tag{B.5}$$

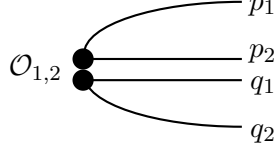


Figure 5.

The action-related operator U_9

From the Ward identity (A.9) we obtain

$$G_{\mathcal{A}}^{(4)} = -\frac{4f_0^2}{NB(p)} + \frac{2f_0}{N}X^{(o)}, \quad (\text{B.6})$$

where the coefficient $X^{(o)}$ is the bare version of (2.54):

$$X^{(o)} = \frac{1}{B(p)B(0)} \left\{ \frac{1}{B(p)} \frac{\partial B(p)}{\partial m^2} + \frac{1}{p_1^2 + m^2} + \frac{1}{p_2^2 + m^2} + \frac{1}{q_1^2 + m^2} + \frac{1}{q_2^2 + m^2} \right\} = X + \mathcal{O}(\varepsilon). \quad (\text{B.7})$$

Subtracting the U_8 part we get

$$G_9^{(4)} = \frac{2f_0}{N}X^{(o)} \quad (\text{B.8})$$

and after renormalization (2.51).

The operators U_1, U_2

There are two types of contributions to the 4-point functions $G_1^{(4)}$ and $G_2^{(4)}$ (recall these are correlators of the operators (A.15)): either the four external legs are attached to the four legs of the local operators, or all four external legs are coupled to one pair of operator legs and the other pair of operator legs are contracted with each other. The latter type of contributions are proportional to the action-related 4-point function $G_{\mathcal{A}}^{(4)}$. Indicating the first type of contributions by a hat, we have

$$G_1^{(4)} = \hat{G}_1^{(4)} - \frac{m^2}{2}G_{\mathcal{A}}^{(4)}, \quad G_2^{(4)} = \hat{G}_2^{(4)} - \frac{m^2}{2D}G_{\mathcal{A}}^{(4)}. \quad (\text{B.9})$$

The “hatted” contributions are constructed from the building blocks shown in figures 5-7

$$\hat{G}_1^{(4)} = \frac{f_0^3}{4N} T_{\mu\mu}(p_1, p_2) T_{\nu\nu}(q_1, q_2), \quad (\text{B.10})$$

$$\hat{G}_2^{(4)} = \frac{f_0^3}{4N} T_{\mu\nu}(p_1, p_2) T_{\mu\nu}(q_1, q_2), \quad (\text{B.11})$$

where

$$-T_{\mu\nu}(p_1, p_2) = p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} + \frac{2}{B(p)}(A_0\delta_{\mu\nu} + B_0p_{\mu}p_{\nu}). \quad (\text{B.12})$$

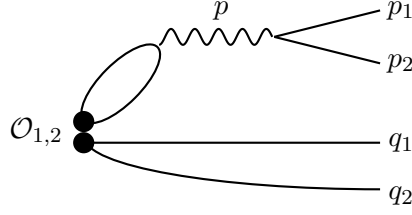


Figure 6.

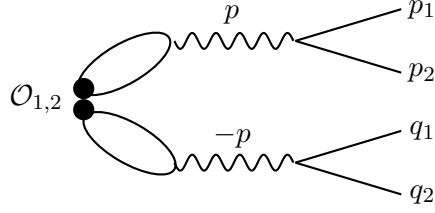


Figure 7.

The p -dependent scalars A_0 , B_0 are defined by the regularized loop integral

$$\int \frac{d^D q}{(2\pi)^D} \frac{q_\mu (q+p)_\nu}{(q^2 + m^2)[m^2 + (q+p)^2]} = A_0 \delta_{\mu\nu} + B_0 p_\mu p_\nu \quad (\text{B.13})$$

and are explicitly given as

$$\begin{aligned} A_0 &= \frac{1}{2(D-1)f_0} - \frac{m^2 + p^2/4}{D-1} B(p), \\ p^2 B_0 &= \frac{D-2}{2(D-1)f_0} + \left[\frac{m^2}{D-1} + \frac{p^2(2-D)}{4(D-1)} \right] B(p). \end{aligned} \quad (\text{B.14})$$

A_0 is divergent but B_0 has a finite $D \rightarrow 2$ limit:

$$B_0 = b_0 + \mathcal{O}(\varepsilon), \quad p^2 b_0 = -\frac{1}{4\pi} + m^2 b(p). \quad (\text{B.15})$$

Using

$$DA_0 + p^2 B_0 = \frac{1}{f_0} - (m^2 + p^2/2) B(p), \quad (\text{B.16})$$

$\hat{G}_i^{(4)}$ can be simplified:

$$\begin{aligned} \hat{G}_i^{(4)} &= \frac{f_0^3}{N} \left(m^2 + \frac{p_1^2 + p_2^2}{2} - \frac{1}{f_0 B(p)} \right) \left(m^2 + \frac{q_1^2 + q_2^2}{2} - \frac{1}{f_0 B(p)} \right) \\ &= \frac{f_0}{NB^2(p)} - \frac{2m^2 f_0^2}{NB(p)} + G_3^{(4)} + G_4^{(4)}. \end{aligned} \quad (\text{B.17})$$

Comparing the last line and the definition in (2.13) we see that the 4-point function of U_1 simplifies to

$$G_1^{(4)} = \frac{f_0}{NB^2(p)} - \frac{m^2 f_0}{N} X^{(o)}. \quad (\text{B.18})$$

The renormalized version is obtained by the substitution $f_0 \rightarrow f$ and replacing the rest by its finite $D \rightarrow 2$ limit as given in (2.43).

There is an alternative representation of $\hat{G}_1^{(4)}$ which will be useful later. Going back to the definition (B.10) we can write

$$\hat{G}_1^{(4)} = \frac{f_0^3}{4N} \left\{ \frac{4D^2 A_0^2}{B^2(p)} + \frac{4DA_0}{B(p)} \left[p_1 p_2 + q_1 q_2 + \frac{2B_0 p^2}{B(p)} \right] + \mathcal{F}_1 \right\} \quad (\text{B.19})$$

where

$$\mathcal{F}_1 = \frac{4B_0^2(p^2)^2}{B^2(p)} + \frac{4B_0 p^2}{B(p)} (p_1 p_2 + q_1 q_2) + 4(p_1 p_2)(q_1 q_2). \quad (\text{B.20})$$

Similarly, from (B.11) we calculate

$$\hat{G}_2^{(4)} = \frac{f_0^3}{4N} \left\{ \frac{4DA_0^2}{B^2(p)} + \frac{4A_0}{B(p)} \left[p_1 p_2 + q_1 q_2 + \frac{2B_0 p^2}{B(p)} \right] + \mathcal{F}_2 \right\} \quad (\text{B.21})$$

with

$$\mathcal{F}_2 = \frac{4B_0^2(p^2)^2}{B^2(p)} + \frac{4B_0}{B(p)} [(p_1 p)(p_2 p) + (q_1 p)(q_2 p)] + 2(p_1 q_1)(p_2 q_2) + 2(p_1 q_2)(p_2 q_1). \quad (\text{B.22})$$

To compute the 4-point function of U_2 we first observe that the action-related terms cancel and we can start from

$$G_2^{(4)} = \hat{G}_1^{(4)} - D\hat{G}_2^{(4)}. \quad (\text{B.23})$$

After some algebra we see that the divergent A_0 -dependent terms also cancel and we are left with

$$G_2^{(4)} = \frac{f_0^3}{N} \left\{ - (D-1) \left(\frac{B_0 p^2}{B(p)} \right)^2 + \frac{B_0 p^2}{B(p)} \left[p_1 p_2 + q_1 q_2 - \frac{D}{p^2} (p_1 p)(p_2 p) - \frac{D}{p^2} (q_1 p)(q_2 p) \right] + (p_1 p_2)(q_1 q_2) - \frac{D}{2} (p_1 q_1)(p_2 q_2) - \frac{D}{2} (p_1 q_2)(p_2 q_1) \right\}. \quad (\text{B.24})$$

The renormalized 4-point function (2.44) is obtained from this in the $D \rightarrow 2$ limit by making the substitutions $f_0 \rightarrow f$ and

$$\frac{B_0 p^2}{B(p)} \longrightarrow \frac{b_0 p^2}{b(p)} = m^2 - \frac{1}{4\pi b(p)} = Q. \quad (\text{B.25})$$

The tensor operator U_7

The essential part of U_7 is the operator

$$\frac{N}{f_0} \sum_{\mu=1}^D (\partial_\mu S \cdot \partial_\mu S)^2 \quad (\text{B.26})$$

and we will denote its 4-point function by $G_7^{(4)}$. Very similarly to the previously discussed cases, we can separate this 4-point function into two parts:

$$G_7^{(4)} = \hat{G}_7^{(4)} - \frac{4m^2}{D} G_{\mathcal{A}}^{(4)}, \quad (\text{B.27})$$

where

$$\hat{G}_7^{(4)} = \frac{8f_0^3}{N} \sum_{\mu=1}^D \left(p_{1\mu} p_{2\mu} + \frac{A_0}{B(p)} + \frac{B_0 p_\mu^2}{B(p)} \right) \left(q_{1\mu} q_{2\mu} + \frac{A_0}{B(p)} + \frac{B_0 p_\mu^2}{B(p)} \right). \quad (\text{B.28})$$

This can be simplified to

$$\hat{G}_7^{(4)} = \frac{8f_0^3}{N} \left\{ \frac{DA_0^2}{B^2(p)} + \frac{A_0}{B(p)} \left[p_1 p_2 + q_1 q_2 + \frac{2B_0 p^2}{B(p)} \right] + \mathcal{F}_7 \right\} \quad (\text{B.29})$$

where

$$\mathcal{F}_7 = \sum_{\mu=1}^D \left(p_{1\mu} p_{2\mu} + \frac{B_0 p_\mu^2}{B(p)} \right) \left(q_{1\mu} q_{2\mu} + \frac{B_0 p_\mu^2}{B(p)} \right). \quad (\text{B.30})$$

Using (2.10) we notice that action-dependent terms again cancel and we can write

$$G_7^{(4)} = \hat{G}_7^{(4)} - \frac{8}{D+2} \hat{G}_1^{(4)} - \frac{16}{D+2} \hat{G}_2^{(4)}. \quad (\text{B.31})$$

The divergent, A_0 -dependent terms also cancel here and we are left with

$$G_7^{(4)} = \frac{f_0^3}{N} \left(8\mathcal{F}_7 - \frac{2}{D+2} \mathcal{F}_1 - \frac{4}{D+2} \mathcal{F}_2 \right). \quad (\text{B.32})$$

The final result (2.49) is obtained from this last formula by making the replacement $f_0 \rightarrow f$ and writing the $D \rightarrow 2$ limit of the rest as a quadratic polynomial in Q .

The tensor operator U_6

Here we will use the following representation [7]¹⁰ of the operator U_6 :

$$U_6 = Q_6 - \frac{6}{D+2} \left(U_5 - 4U_4 + \frac{4N}{f_0} \mathcal{O}_1 \right). \quad (\text{B.33})$$

Here Q_6 , defined in (A.34), is the main part whose correlation functions we will denote by the dotted index $\dot{6}$. Using the fact that U_5 is a pure contact term from the above representation we have

$$G_6^{(4)} = G_{\dot{6}}^{(4)} - \frac{24}{D+2} \left(G_{\dot{1}}^{(4)} - G_4^{(4)} \right). \quad (\text{B.34})$$

There are several contributions to the four-point correlation function $G_6^{(4)}$. The corresponding Feynman diagrams are shown in figures 8-11.

¹⁰This follows from formulae (A.10) and (A.34) of that paper in the large N limit.

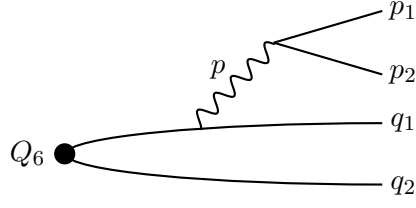


Figure 8.

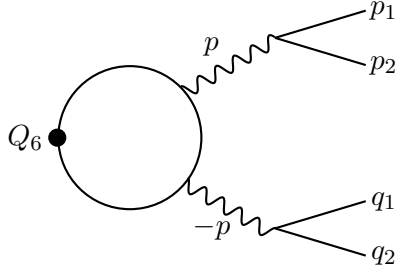


Figure 9.

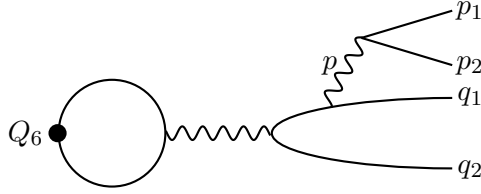


Figure 10.

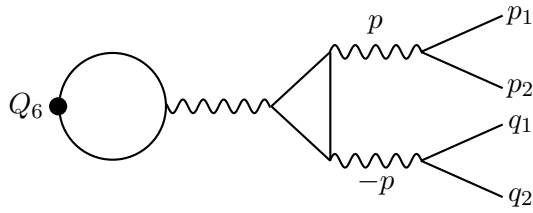


Figure 11.

The sum of the contributions of the Feynman diagrams shown in figure 8 is

$$-\frac{4f_0^2}{NB(p)} \sum_{\mu=1}^D \left\{ \frac{p_{1\mu}^4}{p_1^2 + m^2} + \frac{p_{2\mu}^4}{p_2^2 + m^2} + \frac{q_{1\mu}^4}{q_1^2 + m^2} + \frac{q_{2\mu}^4}{q_2^2 + m^2} \right\}. \quad (\text{B.35})$$

The 9 type contribution is

$$\frac{8f_0^2}{NB^2(p)} \int \frac{d^D q}{(2\pi)^D} \sum_{\mu=1}^D \frac{q_\mu^4}{(q^2 + m^2)^2 [m^2 + (q+p)^2]}. \quad (\text{B.36})$$

The sum of the 10 type contributions is

$$\frac{4f_0^2}{NB(p)B(0)} \left\{ \frac{1}{p_1^2 + m^2} + \frac{1}{p_2^2 + m^2} + \frac{1}{q_1^2 + m^2} + \frac{1}{q_2^2 + m^2} \right\} \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2} \quad (\text{B.37})$$

and finally the 11 type contribution is

$$\frac{4f_0^2}{NB^2(p)B(0)} \frac{\partial B(p)}{\partial m^2} \left(\sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2} \right). \quad (\text{B.38})$$

Putting together all contributions we have

$$\begin{aligned} G_6^{(4)} = & -\frac{4f_0^2}{NB(p)} \sum_{\mu=1}^D \left\{ \frac{p_{1\mu}^4}{p_1^2 + m^2} + \frac{p_{2\mu}^4}{p_2^2 + m^2} + \frac{q_{1\mu}^4}{q_1^2 + m^2} + \frac{q_{2\mu}^4}{q_2^2 + m^2} \right\} \\ & + \frac{4f_0^2}{N} X^{(o)} \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2} \\ & + \frac{8f_0^2}{NB^2(p)} \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2 [m^2 + (q+p)^2]}. \end{aligned} \quad (\text{B.39})$$

Using our earlier results we have

$$G_1^{(4)} - G_4^{(4)} = \frac{f_0}{NB^2(p)} + G_3^{(4)} - \frac{m^2 f_0}{N} X^{(o)} \quad (\text{B.40})$$

and finally

$$\begin{aligned} G_6^{(4)} = & -\frac{4f_0^2}{NB(p)} \sum_{\mu=1}^D \left\{ \frac{p_{1\mu}^4}{p_1^2 + m^2} + \frac{p_{2\mu}^4}{p_2^2 + m^2} + \frac{q_{1\mu}^4}{q_1^2 + m^2} + \frac{q_{2\mu}^4}{q_2^2 + m^2} \right\} \\ & + \frac{12f_0^2}{(D+2)NB(p)} (p_1^2 + p_2^2 + q_1^2 + q_2^2) \\ & + \frac{4f_0^2}{N} X^{(o)} \left\{ \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2} + \frac{6}{D+2} \frac{m^2}{f_0} \right\} \\ & + \frac{8f_0^2}{NB^2(p)} \left\{ \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2 [m^2 + (q+p)^2]} - \frac{3}{(D+2)f_0} \right\}. \end{aligned} \quad (\text{B.41})$$

The coefficient of $X^{(o)}$ can be simplified if we evaluate

$$\int \frac{d^D q}{(2\pi)^D} \frac{q_\alpha q_\beta q_\gamma q_\delta}{(q^2 + m^2)^2} = \Psi(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \quad (\text{B.42})$$

Ψ can be calculated by considering the contraction

$$D(D+2)\Psi = \int \frac{d^D q}{(2\pi)^D} \frac{(q^2)^2}{(q^2 + m^2)^2} = -\frac{2m^2}{f_0} + m^4 B(0). \quad (\text{B.43})$$

This gives

$$\int \frac{d^D q}{(2\pi)^D} \sum_{\mu=1}^D \frac{q_\mu^4}{(q^2 + m^2)^2} = 3D\Psi = -\frac{6m^2}{(D+2)f_0} + \frac{3m^4 B(0)}{D+2}. \quad (\text{B.44})$$

The four-point function is now simplified to

$$\begin{aligned} G_6^{(4)} = & -\frac{4f_0^2}{NB(p)} \sum_{\mu=1}^D \left\{ \frac{p_{1\mu}^4}{p_1^2 + m^2} + \frac{p_{2\mu}^4}{p_2^2 + m^2} + \frac{q_{1\mu}^4}{q_1^2 + m^2} + \frac{q_{2\mu}^4}{q_2^2 + m^2} \right\} \\ & + \frac{12f_0^2}{(D+2)NB(p)} (p_1^2 + p_2^2 + q_1^2 + q_2^2) + \frac{12m^4 f_0^2 B(0)}{(D+2)N} X^{(o)} \\ & + \frac{8f_0^2}{NB^2(p)} Y^{(o)}, \end{aligned} \quad (\text{B.45})$$

where we introduced a second bare coefficient

$$Y^{(o)} = \sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2 [m^2 + (q+p)^2]} - \frac{3}{(D+2)f_0} = Y + \mathcal{O}(\varepsilon). \quad (\text{B.46})$$

The formula (B.45) is now ready for renormalization and for the renormalized four-point function we get (2.48) of section 2.

It remains to calculate the finite part Y . We proceed similarly to the evaluation of the coefficient of $X^{(o)}$ above but here the calculation is more difficult because of the momentum dependence. We start by defining

$$\begin{aligned} \int \frac{d^D q}{(2\pi)^D} \frac{q_\alpha q_\beta q_\gamma q_\delta}{(q^2 + m^2)^2 [m^2 + (q+p)^2]} = & \Omega_1 (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \\ & + \Omega_2 (\delta_{\alpha\beta} p_\gamma p_\delta + \delta_{\alpha\gamma} p_\beta p_\delta + \delta_{\alpha\delta} p_\beta p_\gamma + p_\alpha p_\beta \delta_{\gamma\delta} + p_\alpha p_\gamma \delta_{\beta\delta} + p_\alpha p_\delta \delta_{\beta\gamma}) \\ & + \Omega_3 p_\alpha p_\beta p_\gamma p_\delta. \end{aligned} \quad (\text{B.47})$$

What we need is

$$\sum_{\mu=1}^D \int \frac{d^D q}{(2\pi)^D} \frac{q_\mu^4}{(q^2 + m^2)^2 [m^2 + (q+p)^2]} = 3D\Omega_1 + 6\Omega_2 p^2 + \Omega_3 \sum_{\mu=1}^D p_\mu^4 \quad (\text{B.48})$$

and we expect that

$$\overline{\Omega} = 3D\Omega_1 + 6\Omega_2 p^2 - \frac{3}{(D+2)f_0} = \overline{\omega} + \mathcal{O}(\varepsilon), \quad \Omega_3 = \omega_3 + \mathcal{O}(\varepsilon). \quad (\text{B.49})$$

Anticipating the finiteness of $\overline{\omega}$ and ω_3 we can write

$$Y = \overline{\omega} + \omega_3 p^4. \quad (\text{B.50})$$

For this calculation we have to introduce, in addition to (B.47), the following hierarchy of tensor integrals ($r = 0, 1$):

$$\begin{aligned}
\int \frac{d^D q}{(2\pi)^D} \frac{q_\alpha q_\beta q_\gamma}{(q^2 + m^2)^{(1+r)}[m^2 + (q+p)^2]} &= K_r(\delta_{\alpha\beta} p_\gamma + \delta_{\alpha\gamma} p_\beta + \delta_{\beta\gamma} p_\alpha) \\
&\quad + N_r p_\alpha p_\beta p_\gamma, \\
\int \frac{d^D q}{(2\pi)^D} \frac{q_\alpha q_\beta}{(q^2 + m^2)^{(1+r)}[m^2 + (q+p)^2]} &= H_r \delta_{\alpha\beta} + M_r p_\alpha p_\beta, \\
\int \frac{d^D q}{(2\pi)^D} \frac{q_\alpha}{(q^2 + m^2)^{(1+r)}[m^2 + (q+p)^2]} &= \xi_r p_\alpha.
\end{aligned} \tag{B.51}$$

By making contraction of the indices with Kronecker deltas and the momentum vector p_μ we find the following simple relations among the tensor integrals:

$$\begin{aligned}
(D+2)\Omega_1 + p^2\Omega_2 &= H_0 - m^2 H_1, & (D+4)\Omega_2 + p^2\Omega_3 &= M_0 - m^2 M_1, \\
2\Omega_1 + 2p^2\Omega_2 &= -K_0 - p^2 K_1, & 6\Omega_2 + 2p^2\Omega_3 &= -N_0 - p^2 N_1, \\
(D+2)K_0 + p^2 N_0 &= -1/f_0 - m^2 \xi_0, & (D+2)K_1 + p^2 N_1 &= \xi_0 - m^2 \xi_1, \\
2p^2 K_0 &= -p^2 H_0, & 4K_0 + 2p^2 N_0 &= -1/f_0 - p^2 M_0, \\
2p^2 K_1 &= (1/f_0 - m^2 B(0))/D - H_0 - p^2 H_1, & 4K_1 + 2p^2 N_1 &= -M_0 - p^2 M_1, \\
DH_0 + p^2 M_0 &= 1/f_0 - m^2 B(p), & DH_1 + p^2 M_1 &= B(p) + (m^2/2)(\partial B(p)/\partial m^2), \\
2H_0 + 2p^2 M_0 &= 1/f_0 - p^2 \xi_0, & 2H_1 + 2p^2 M_1 &= -\xi_0 - p^2 \xi_1, \\
2p^2 \xi_0 &= -p^2 B(p), & 2p^2 \xi_1 &= B(0) - B(p) + (p^2/2)(\partial B(p)/\partial m^2).
\end{aligned} \tag{B.52}$$

We have solved this overdetermined system of linear relations using Mathematica. The results for Ω_i are too bulky to be reproduced here. In the limit $D \rightarrow 2$ the formulas simplify enormously and we find

$$\overline{\omega} = -3m^2 b(p) + \frac{3p^2}{4} \left(m^2 + \frac{p^2}{4} \right) \frac{\partial b(p)}{\partial m^2} + \frac{3}{32\pi} \left(5 + \frac{p^2}{m^2} \right), \tag{B.53}$$

$$(p^2)^2 \omega_3 = 2m^2 b(p) - \left(\frac{m^4}{2} + m^2 p^2 + \frac{(p^2)^2}{4} \right) \frac{\partial b(p)}{\partial m^2} - \frac{1}{8\pi} \left(5 + \frac{p^2}{m^2} \right). \tag{B.54}$$

C Lattice integrals

First note that the continuum limit of the lattice integral $B_{\text{latt}}(p)$ in eq. (4.15) is given by $b(p)$ defined in eq. (2.52) which can be expressed analytically as

$$b(p) = b(p, m) = \frac{1}{2\pi\sqrt{p^2(p^2 + 4m^2)}} \ln \left(\frac{\sqrt{p^2 + 4m^2} + \sqrt{p^2}}{\sqrt{p^2 + 4m^2} - \sqrt{p^2}} \right), \tag{C.1}$$

where the explicit reference to the second argument m will be used later in this appendix. $b(p)$ is analytic in p^2 with a cut from $-\infty$ to $-4m^2$. Also note that $b(p) \neq 0$ for all p^2 and

the properties:

$$b(p) \sim \frac{\ln(p^2/m^2)}{2\pi p^2} \quad \text{for } p^2 \rightarrow \infty, \quad (\text{C.2})$$

$$b(0) = \frac{1}{4\pi m^2}. \quad (\text{C.3})$$

Further we have

$$\frac{\partial}{\partial m^2} b(p) = -\frac{2}{p^2 + 4m^2} \left[b(p) + \frac{1}{4\pi m^2} \right], \quad (\text{C.4})$$

$$\frac{\partial}{\partial p_\mu} b(p) = -\frac{2p_\mu}{p^2(p^2 + 4m^2)} \left[(p^2 + 2m^2)b(p) - \frac{1}{2\pi} \right]. \quad (\text{C.5})$$

In order to work out the leading lattice artifacts of $B_{\text{latt}}(p)$ it is convenient to consider the equivalent representation: ¹¹

$$\begin{aligned} B_{\text{latt}}(p) &= a^2 \int_0^\infty dt_1 dt_2 e^{-(t_1+t_2)(a^2 m_0^2+4)} \prod_{\mu=1}^2 I_0 \left(2\sqrt{(t_1+t_2)^2 - t_1 t_2 a^2 \hat{p}_\mu^2} \right) \\ &= a^2 \int_0^\infty dt \int_0^1 dx t e^{-t(a^2 m_0^2+4)} \prod_{\mu=1}^2 I_0(2t S_\mu(x, p)), \end{aligned} \quad (\text{C.6})$$

where

$$S_\mu(x, p) \equiv \sqrt{1 - x(1-x)a^2 \hat{p}_\mu^2}, \quad (\text{C.7})$$

and I_r denote the modified Bessel functions. For $p = 0$ we have simply:

$$B_{\text{latt}}(0) = a^2 \int_0^\infty dt t e^{-a^2 m_0^2 t} [e^{-2t} I_0(2t)]^2. \quad (\text{C.8})$$

Noting for large t :

$$e^{-2t} I_r(2t) = \frac{1}{\sqrt{4\pi t}} \left[1 + \frac{1-4r^2}{16t} + \mathcal{O}(t^{-2}) \right] \quad (\text{C.9})$$

and splitting the range of the t -integration in (C.8) into parts $[0, 1]$ and $[1, \infty]$ we obtain:

$$B_{\text{latt}}(0) = \frac{1}{4\pi m_0^2} + a^2 \left[c_1 - \frac{1}{32\pi} \ln(a^2 m_0^2) \right] + \mathcal{O}(a^4 m_0^2 \ln(a^2 m_0^2)), \quad (\text{C.10})$$

with

$$c_1 = -\frac{1}{4\pi} + \sum_{k=1}^4 c_1^{(k)}, \quad (\text{C.11})$$

$$c_1^{(1)} = \int_0^1 dt t [e^{-2t} I_0(2t)]^2, \quad (\text{C.12})$$

$$c_1^{(2)} = \int_1^\infty dt \left\{ t [e^{-2t} I_0(2t)]^2 - \frac{1}{4\pi} \left(1 + \frac{1}{8t} \right) \right\}, \quad (\text{C.13})$$

$$c_1^{(3)} = -\frac{1}{32\pi} \int_0^1 dt t^{-1} (1 - e^{-t}), \quad (\text{C.14})$$

$$c_1^{(4)} = \frac{1}{32\pi} \int_1^\infty dt t^{-1} e^{-t}. \quad (\text{C.15})$$

¹¹before performing the q_μ integrations we shift the variables $q_\mu \rightarrow q_\mu + \alpha_\mu$ with $\sin(a\alpha_\mu) = -t_2 \sin(ap_\mu) T_\mu^{-1}$, $\cos(a\alpha_\mu) = [t_1 + t_2 \cos(ap_\mu)] T_\mu^{-1}$, $T_\mu = \sqrt{(t_1 + t_2)^2 - t_1 t_2 a^2 \hat{p}_\mu^2}$.

c_1 can in fact be determined analytically as

$$c_1 = \frac{1}{16\pi} \left[\frac{5}{2} \ln(2) - 1 \right], \quad (\text{C.16})$$

which can easily be checked numerically.

Proceeding as for the case $p = 0$ above:

$$B_{\text{latt}}(p) = a^2 c_1^{(1)} + a^2 \int_1^\infty dt \int_0^1 dx te^{-t(a^2 m_0^2 + 4)} \prod_{\mu=1}^2 I_0(2tS_\mu(x, p)) + \mathcal{O}(a^4) \quad (\text{C.17})$$

$$= a^2 \left[c_1^{(1)} + c_1^{(2)} \right] + a^2 \frac{1}{4\pi} \int_1^\infty dt \int_0^1 dx \frac{e^{-t[a^2 m_0^2 + 4 - 2 \sum_{\nu=1}^2 S_\nu(x, p)]}}{\prod_{\mu=1}^2 S_\mu(x, p)^{1/2}} \left[1 + \frac{1}{8t} \right] + \mathcal{O}(a^4) \quad (\text{C.18})$$

$$= a^2 \left[c_1^{(1)} + c_1^{(2)} \right] + \frac{a^2}{4\pi} \int_1^\infty dt \int_0^1 dx e^{-ta^2[m_0^2 + x(1-x)\hat{p}^2]} \times \left\{ 1 + \frac{1}{8t} + \frac{1}{4}x(1-x)a^2\hat{p}^2 - \frac{t}{4}x^2(1-x)^2a^4\hat{p}^4 \right\} + \mathcal{O}(a^4) \quad (\text{C.19})$$

$$= b(\hat{p}, m_0) + a^2 \left[c_1 + \sum_{k=1}^3 \beta_k \right] + \mathcal{O}(a^4), \quad (\text{C.20})$$

with

$$\beta_1 = \frac{1}{16\pi} \int_0^1 dx \frac{x(1-x)p^2}{[m^2 + x(1-x)p^2]} \quad (\text{C.21})$$

$$= \frac{1}{16\pi} - \frac{1}{4}m^2b(p), \quad (\text{C.22})$$

$$\beta_2 = -\frac{1}{16\pi} \int_0^1 dx \frac{x^2(1-x)^2p^4}{[m^2 + x(1-x)p^2]^2} \quad (\text{C.23})$$

$$= -\frac{p^4}{4(p^2)^2} \left[\frac{1}{4\pi} - 2m^2b(p) - m^4 \frac{\partial}{\partial m^2} b(p) \right], \quad (\text{C.24})$$

$$\beta_3 = -\frac{1}{32\pi} \int_0^1 dx \ln[a^2m^2 + x(1-x)a^2p^2] \quad (\text{C.25})$$

$$= -\frac{1}{16} \left[(p^2 + 4m^2)b(p) + \frac{1}{2\pi} \ln(a^2m^2) - \frac{1}{\pi} \right]. \quad (\text{C.26})$$

Putting the above results together and expanding m_0 and \hat{p} in m and p respectively (i.e. in a) we obtain (4.16) with

$$b_1(p) = \frac{m^4}{12} \frac{\partial}{\partial m^2} b(p) - \sum_\mu \frac{p_\mu^3}{24} \frac{\partial}{\partial p_\mu} b(p) + c_1 + \beta_1 + \beta_2 + \beta_3. \quad (\text{C.27})$$

Using eqs. (C.4), (C.5) and inserting the resulting $b_1(p)$ into eq. (4.23) one indeed obtains eq. (4.24).

For the tadpole (which we need below) one gets

$$J(m_0) \equiv \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{1}{m_0^2 + \hat{q}^2} \quad (\text{C.28})$$

$$= -\frac{1}{4\pi} \ln(a^2 m_0^2) + c_2 + a^2 m_0^2 \left[\frac{1}{32\pi} \ln(a^2 m_0^2) - \frac{1}{32\pi} - c_1 \right] + \mathcal{O}(a^4) , \quad (\text{C.29})$$

where

$$c_2 = \frac{5}{4\pi} \ln(2) = 8c_1 + \frac{1}{2\pi} . \quad (\text{C.30})$$

Next we consider the integrals occurring in the 4-point function of the mixed action.

First

$$\begin{aligned} h_\mu(p) &= \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{\hat{q}_\mu \widehat{(p+q)}_\mu}{(m_0^2 + \hat{q}^2) \left[m_0^2 + \widehat{(p+q)}^2 \right]} \\ &= 2 \cos\left(\frac{ap_\mu}{2}\right) \int_0^\infty dt \int_0^1 dx te^{-t(a^2 m_0^2 + 4)} I_0(2tS_{\overline{\mu}}(x, p)) \times \\ &\quad \times \left[I_0(2tS_\mu(x, p)) - \frac{I_1(2tS_\mu(x, p))}{S_\mu(x, p)} \right] , \end{aligned} \quad (\text{C.31})$$

where $\overline{1} = 2, \overline{2} = 1$. The expansion for $h_\mu(0)$ to order a^2 is immediately obtained from eq. (C.29):

$$\begin{aligned} h_\mu(0) &= \frac{1}{2} \left[J(m_0) + m_0^2 \frac{\partial}{\partial m_0^2} J(m_0) \right] \\ &= -\frac{1}{8\pi} \ln(a^2 m_0^2) + \frac{c_2}{2} - \frac{1}{8\pi} + a^2 m_0^2 \left[\frac{1}{32\pi} \ln(a^2 m_0^2) - \frac{1}{64\pi} - c_1 \right] + \mathcal{O}(a^4) . \end{aligned} \quad (\text{C.32})$$

Actually for our purposes we only need $h_\mu(p)$ to leading order

$$h_\mu(p) = 2h_\mu^{(1)}(p) - \hat{p}_\mu^2 h_\mu^{(2)}(p) + \mathcal{O}(a^2) , \quad (\text{C.33})$$

$$\begin{aligned} h_\mu^{(1)}(p) &= \int_0^\infty dt \int_0^1 dx te^{-t(a^2 m_0^2 + 4)} I_0(2tS_{\overline{\mu}}(x, p)) \\ &\quad \times [I_0(2tS_\mu(x, p)) - I_1(2tS_\mu(x, p))] , \end{aligned} \quad (\text{C.34})$$

$$h_\mu^{(2)}(p) = a^2 \int_0^\infty dt \int_0^1 dx tx(1-x)e^{-t(a^2 m_0^2 + 4)} I_0(2tS_{\overline{\mu}}(x, p)) I_1(2tS_\mu(x, p)) . \quad (\text{C.35})$$

From these representations we can deduce the leading terms

$$h_\mu^{(1)}(p) = \frac{c_2}{4} - \frac{1}{16\pi} + 2\beta_3 + \mathcal{O}(a^2) , \quad (\text{C.36})$$

$$h_\mu^{(2)}(p) = \frac{4}{p^2} \beta_1 + \mathcal{O}(a^2) , \quad (\text{C.37})$$

where the β_k are defined in eqs. (C.21)-(C.25). Adding the terms we get

$$h_\mu(p) = \frac{z}{2} + \frac{1}{8\pi} - \frac{1}{4}(p^2 + 4m^2)b(p) + \frac{p_\mu^2}{p^2}b(p)Q + \mathcal{O}(a^2) , \quad (\text{C.38})$$

with Q, z defined in eqs. (2.53), (1.19) respectively.

Finally for

$$h_{\mu\nu}(p) = \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{\widehat{q}_\mu(p+q)_\mu \widehat{q}_\nu(p+q)_\nu}{(m_0^2 + \widehat{q}^2) \left[m_0^2 + (\widehat{p+q})^2 \right]} \quad (\text{C.39})$$

appearing in the definition of $\widetilde{\Delta}_{\mu\nu}(p)$ (cf. [8], app. B) and which we need here only for $p = 0, m_0 = 0$, we have

$$h_{\mu\nu}(0)|_{m_0=0} = \int_{-\pi/a}^{\pi/a} \frac{d^2 q}{(2\pi)^2} \frac{\widehat{q}_\mu^2 \widehat{q}_\nu^2}{(\widehat{q}^2)^2} = \frac{1}{2\pi a^2} [(1 - \delta_{\mu\nu}) + (\pi - 1)\delta_{\mu\nu}] . \quad (\text{C.40})$$

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